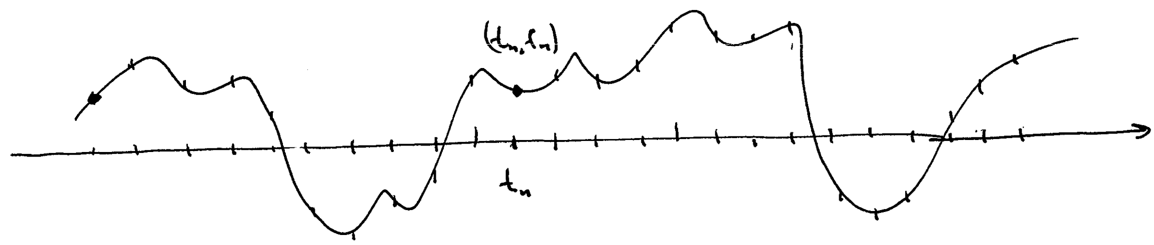
The discrete wavelet transform

Question: how to compute the scaling and wavelet coefficients?

$$f_J(t) = \sum_{j=j_0}^{J-1} \sum_k w_{j,k} \psi_{j,k}(t) + \sum_k s_{j_0} \varphi_{j_0,k}(t)$$

First: we have to decide a finest scale, J_0 and determine $s_{J,k} = \langle f, \varphi_{J,k} \rangle$.



Usually the signal $f(t)$ is sampled to give a discrete signal $(f_n)_{n=-\infty}^{\infty}$. In the case of the Haar wavelet the $s_{J,k}$ are simply the f_n, but for higher order wavelets the calculation is more involved. Note that the signal must not contain too high frequencies.

Once the $s_{J,k}$ are known, we may proceed recursively:

Assume that $s_{j+1,k} = \langle f, \varphi_{j+1,k} \rangle$ are known.

We may write $f_{j+1} = \sum \langle f, \varphi_{j+1,k} \rangle \varphi_{j+1,k}$ (n.b. notation!)

and we know that $f_{j+1} = f_j + d_j$.

Hence

$$f_{j+1}(t) = \sum_k s_{j+1,k}\,\varphi_{j+1,k}(t)$$

$$= \sum_k s_{j,k}\,\varphi_{j,k}(t) \quad + \sum_k w_{j,k}\,\psi_{j,k} \quad = f_j(t) + d_j(t).$$

We may compute the scalar product with $\varphi_{j,l}$, to get

$$\sum_k s_{j+1,k} \langle \varphi_{j+1,k}, \varphi_{j,l} \rangle = \sum_k s_{j,k} \underbrace{\langle \varphi_{j,k}, \varphi_{j,l} \rangle}_{= \delta_{k,l}} + \sum_k w_{j,k} \underbrace{\langle \psi_{j,k}, \varphi_{j,l} \rangle}_{= 0}$$

Because $\quad \varphi_{j,k} = \sqrt{2} \sum_m h_m \varphi_{j+1,m+2k}$

$\ulcorner$ we check this:

$$\varphi_0 = \varphi(t) = \sum_m h_m \, 2\varphi(2t-m).$$

Then $\quad \varphi_{0,k} = \varphi(t-k) = \sum_m h_m \sqrt{2}\, \sqrt{2}\, \varphi(2t-m-2k)$

$$= \sum_m \sqrt{2}\, h_m \, \varphi_{1,2k+m}(t) \;\lrcorner$$

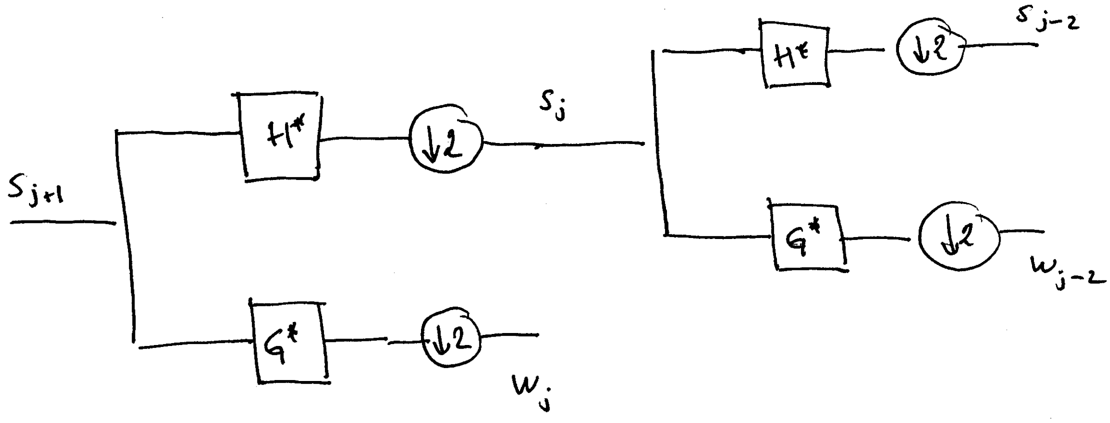
so $\quad \langle \varphi_{j+1,k}, \varphi_{j,l} \rangle = \sqrt{2} \sum_m h_m \underbrace{\langle \varphi_{j+1,k}, \varphi_{j+1,m+2l} \rangle}_{= \delta_{k,2l+m}} = \sqrt{2}\, h_{k-2l}.$

A similar calculation holds for $\langle f_{j+1}, \psi_{j,k} \rangle$, and in summary

$$s_{j,k} = \sqrt{2} \sum_l h_{l-2k}\, s_{j+1,l}$$

$$w_{j,k} = \sqrt{2} \sum_l g_{l-2k}\, s_{j+1,l}$$

This may be expressed with a filter bank:



The inverse calculation is carried out similarly:

Write

$$f_{j+1} = \sum_l s_{j,l}\, \varphi_{j,l} + \sum_l w_{j,l}\, \psi_{j,l}$$

$$= \sqrt{2} \sum_l \sum_m s_{j,l}\, h_m\, \varphi_{j+1,m+2l} + \sqrt{2} \sum_l \sum_m w_{j,l}\, g_m\, \varphi_{j+1,m+2l}$$

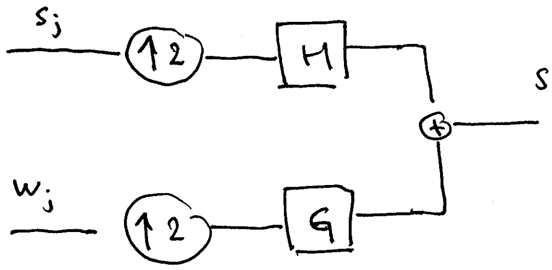
and

$$s_{j+1,k} = \langle f_{j+1}, \varphi_{j+1,k} \rangle =$$

$$= \sqrt{2} \sum_l \sum_m \left(s_{j,l}\, h_m + w_{j,l}\, g_m \right) \langle \varphi_{j+1,m+2l}, \varphi_{j+1,k} \rangle$$

$$= \sqrt{2} \sum_l \left(s_{j,l}\, h_{k-2l} + w_{j,l}\, g_{k-2l} \right)$$

This corresponds to the following filter bank:



(the factors $\sqrt{2}$ may be included in the filters)

Biorthogonal systems

Let $\{V_j\}$ be an MRA, and let

$\{\tilde{V}_j\}$ be a dual MRA.

The scaling function and mother wavelets for $\tilde{V}_j$

satisfy

$$\tilde{\varphi}(t) = 2 \sum \tilde{h}_k \tilde{\varphi}(2t-k)$$
$$\tilde{\psi}(t) = 2 \sum \tilde{g}_k \tilde{\varphi}(2t-k) .$$

The biorthogonality conditions are

$$\langle \varphi_{j,k}, \tilde{\varphi}_{j,l} \rangle = \delta_{k,l}$$
$$\langle \psi_{j,k}, \psi_{j,l} \rangle = \delta_{kl}$$
$$\langle \varphi_{j,k}, \tilde{\psi}_{j,l} \rangle = 0$$
$$\langle \tilde{\varphi}_{j,k}, \psi_{j,l} \rangle = 0$$

At the finest resolution we may then write

$$f_J(t) = \sum_k \langle f, \tilde{\varphi}_{J,k} \rangle \varphi_{J,k}(t) = \sum_k s_{J-1,k} \varphi_{J-1,k} + \sum_k w_{J-1,k} \psi_{J-1,k}$$

$$= \cancel{\sum_k s_{j,k} \varphi_{J-1,k}(t) + \sum_k w_{j,k} \psi_J}$$

and just as in the orthogonal case,

$$f(t) = \sum_{j,k} \langle f, \tilde{\psi}_{j,k} \rangle \psi_{j,k} .$$

Note that in this formula we have both $\tilde{\psi}_{j,k}$ and $\psi_{j,k}$.

Approximation

Assume that the scaling function is such that it reproduces polynomials of order N-1. That means that for each $\alpha = 0, \dots, N-1$,

$$t^\alpha = \sum_k c_k^\alpha \varphi(t-k), \qquad \text{for some coefficients } \{c_k^\alpha\}.$$

It is a standard result from interpolation theory, that if f is differentiable α times, then it can be well approximated by polynomials, ~~of~~ an it is possible to deduce

$$\| f - P_j f \| \le C\, 2^{-j\alpha} \| D^\alpha f \|.$$

Note that if $t^\alpha \in V_j$, then $t^\alpha \perp \tilde{W}_j$, and therefore

$$\langle t^\alpha, \tilde{\psi}_{j,k} \rangle = 0 \quad \text{for every wavelet } \tilde{\psi}_{j,k}, \text{ i.e.,}$$

$$\int t^\alpha \tilde{\psi}_{j,k}(t)\, dt = 0, \qquad \alpha = 0, \dots, N-1.$$

We say that the dual wavelets have N vanishing moments (including 0). But that means in turn that

$$D^\alpha \mathcal{F}(\tilde{\psi})(0) = 0 \quad \text{for} \quad \alpha = 0, \dots, N-1.$$

And because

$$\mathcal{F}(\tilde{\psi})(2\omega) = \mathcal{F}(\tilde{\varphi})(\omega)\ \tilde{G}(\omega) \qquad \text{and} \quad \mathcal{F}(\tilde{\varphi})(0) = 1,$$

we ~~may~~ must have $\tilde{G}(\omega)$ has a zero of order N at $\omega = 0$, which in turn implies that

$$H(\omega) = \left(\frac{e^{-i\omega} + 1}{2} \right)^N Q(\omega), \qquad \text{where } Q \text{ is } 2\pi\text{-periodic}.$$

Two dimensional signal processing

$\mathbb{Z}^2 = \{ (k_x, k_y), \quad k_x = \dots -1, 0, 1, \dots, \quad k_y = \dots -1, 0, 1, \dots \}$

A filter in two dimensions is a map that takes a signal (image), and transforms it:

$$H : f \mapsto g$$

where $f : \mathbb{Z}^2 \to \mathbb{R}$ (or $\mathbb{C}$)
$\qquad g : \mathbb{Z}^2 \to \mathbb{R}$ (or $\mathbb{C}$)

The "shift operator", S^n, $n \in \mathbb{Z}^2$, is defined by $g = S^n f$, $\quad g_k = f_{k-n} \quad \forall k \in \mathbb{Z}^2$.

A Filter is "shift invariant" if

$$H(S^n f) = S^n (Hf) .$$

The impulse response of a filter is

$$h = H\delta ,$$

where $\delta_k = \begin{cases} 1 & \text{when } k = \binom{0}{0} \\ 0 & \text{otherwise} \end{cases}$.

If $f = (f_n)_{n \in \mathbb{Z}^2}$, then $g = Hf$ may be written

$$g = \sum f_n S^n h \underset{\uparrow \atop \text{def}}{=} h * f .$$

The 2-dimensional (discrete) Fourier transform is

$$F(\xi,\eta) = \sum_{k\in\mathbb{Z}^2} f_k\, e^{-i(k_x\xi + k_y\eta)},$$

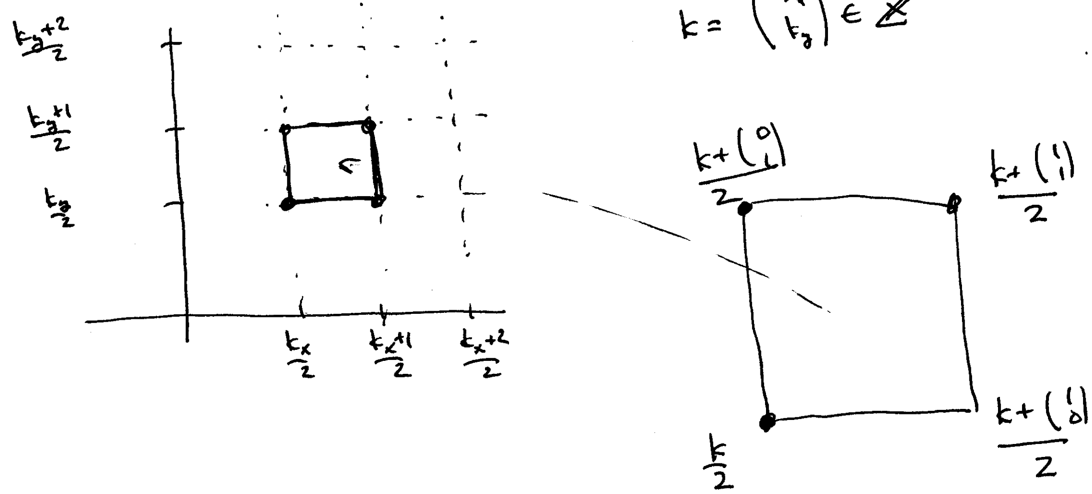
and the convolution theorem is

$$g = h * f \iff G(\xi,\eta) = H(\xi,\eta)\,F(\xi,\eta).$$

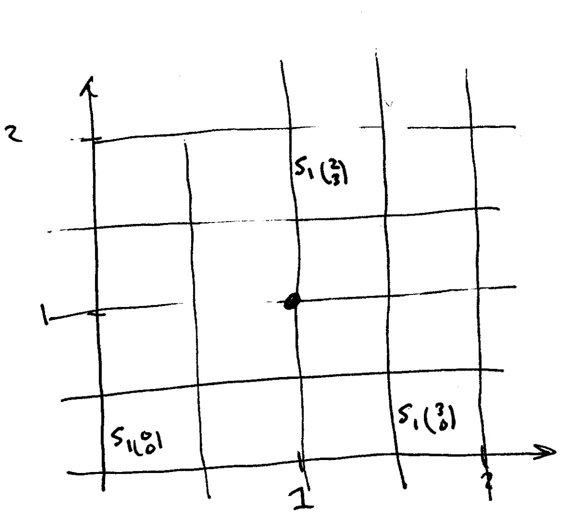
Wavelets in higher dimension

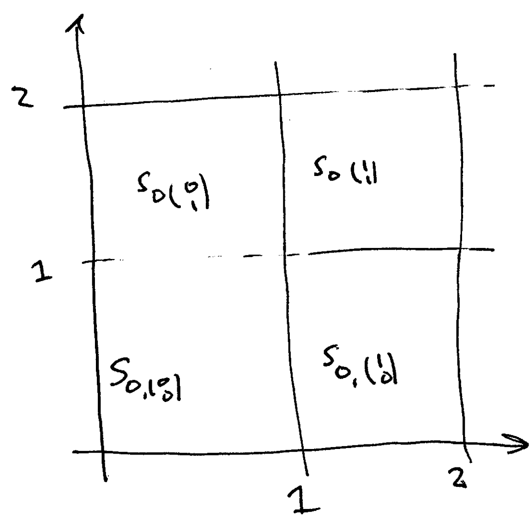
The 2-dimensional Haar system

$k = \begin{pmatrix} k_x \\ k_y \end{pmatrix} \in \mathbb{Z}^2$



Notation





We construct a function

$f_1(x,y)$ that is constant on the indicated squares, with values $s_{1,k}$. We can then reduce the resolution by computing averages:

$$s_{0,k} = \frac{1}{4}\left(s_{1,2k} + s_{1,2k+\binom{1}{0}} + s_{1,2k+\binom{0}{1}} + s_{1,2k+\binom{1}{1}} \right)$$

and similarly we may compute differences (but now there are three different ones):

$$w^H_{0,k} = \frac{1}{4}\left(s_{1,2k} + s_{1,2k+\binom{1}{0}} - s_{1,2k+\binom{0}{1}} - s_{1,2k+\binom{1}{1}} \right)$$

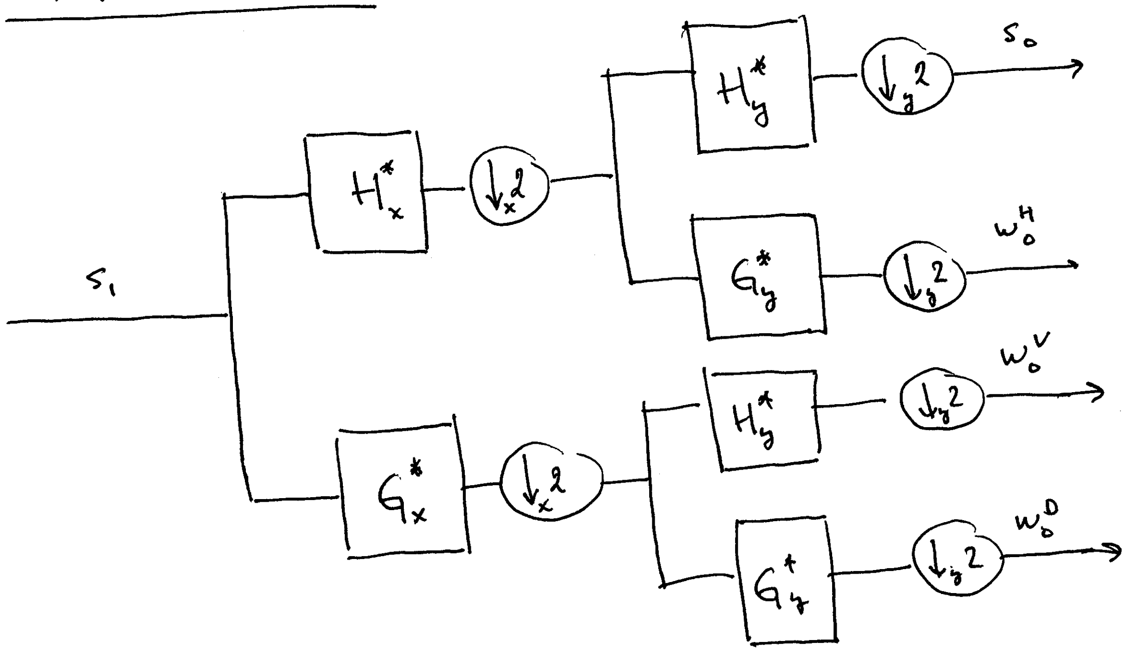
$$w^V_{0,k} = \frac{1}{4}\left(s_{1,2k} - s_{1,2k+\binom{1}{0}} + s_{1,2k+\binom{0}{1}} - s_{1,2k+\binom{1}{1}} \right)$$

$$w^D_{0,k} = \frac{1}{4}\left(s_{1,2k} - s_{1,2k+\binom{1}{0}} - s_{1,2k+\binom{0}{1}} + s_{1,2k+\binom{1}{1}} \right)$$

Here H, V and D stand for "horizontal", "vertical" and "diagonal", and they may be remembered by

$$\begin{array}{|c|c|}\hline + & + \\ \hline + & + \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline - & - \\ \hline + & + \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline - & + \\ \hline + & - \\ \hline \end{array}$$

<u>With filter banks</u>:



Separable scaling functions

For the Haar system,

$$\Phi(x,y) = \phi(x)\,\phi(y)$$

$$\psi^H(x,y) = \phi(x)\,\psi(y)$$

$$\psi^V(x,y) = \psi(x)\,\phi(y)$$

$$\psi^D(x,y) = \psi(x)\,\psi(y).$$

Any 1-dimensional wavelet system may be used to construct a 2-dimensional system in this way.

The scaling equation in this case is

$$\Phi(x,y) = 4 \sum_k h_k\, \Phi(2x-k_x,\, 2y-k_y)$$

$$\psi^\nu(x,y) = 4 \sum_k g_k^\nu\, \Phi(2x-k_x,\, 2y-k_y) \qquad \nu = D, V, H.$$

The MRA is as before:

$$V_j < V_{j+1} \qquad W_j^\nu < V_{j+1},$$

and

$$f_{j+1} = f_j + d_j^H + d_j^V + d_j^D.$$

Multi dimensional Fourier transforms

Notation: $x = (x_1, \dots, x_n)$
$\xi = (\xi_1, \dots, \xi_n)$

If $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ $(k_i \geq 0)$,
let $|k| = k_1 + \dots + k_n$

We define $x^k = x_1^{k_1} \cdot \dots \cdot x_n^{k_n}$

$$\frac{\partial^{|k|}}{\partial x^k} = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdot \dots \cdot \partial x_n^{k_n}}$$

Def Let $f : \mathbb{R}^n \to \mathbb{C}$; then

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x)\, dx .$$

Prop The inverse transform is given by

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \xi} \hat{f}(\xi)\, d\xi .$$

Almost all rules from the Fourier transform in $\mathbb{R}$ apply:

$$f(x,y) \supset \hat{f}(\xi,\eta)$$

[note that here we write f as a function of two arguments, $x, y \in \mathbb{R}$]

$$f(ax, by) \supset \frac{1}{|ab|} \hat{f}\left(\frac{\xi}{a}, \frac{\eta}{b}\right)$$

(for $a, b \in \mathbb{R}$)

$$f(x-a, y-b) \supset e^{-2\pi i (a\xi + b\eta)} \hat{f}(\xi,\eta)$$

$$\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n f(x,y) \supset (2\pi i \xi)^m (2\pi i \eta)^n \hat{f}(\xi,\eta)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) \supset -4\pi^2 (\xi^2 + \eta^2) \hat{f}(\xi,\eta)$$

Distributions in $\mathbb{R}^n$

Def $S(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ is defined by the family of seminorms $\varphi \mapsto \sup_{x \in \mathbb{R}^n} \left| x^\alpha \frac{\partial^\beta}{\partial x^\beta} \varphi(x) \right|$, $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^n$.

Def $\varphi_n \to 0$ in $S(\mathbb{R}^n)$ if for all $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha \frac{\partial^\beta}{\partial x^\beta} \varphi_k \right| \to 0 \quad \text{when } k \to \infty .$$

Def A tempered distribution in $\mathbb{R}^n$ is a complex valued linear functional T,

$T: S(\mathbb{R}^n) \longrightarrow \mathbb{C}$, such that

$T(\varphi_n) \to 0$ when $n \to \infty$ for all sequences $\varphi_n \to 0$ in $S(\mathbb{R}^n)$.

Ex $\delta \otimes \delta$ (this could be written $\delta(x)\delta(y)$)

$$\langle \delta \otimes \delta, \varphi \rangle = \varphi(0,0) .$$

Note this is common notation: if $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ the function $h(x,y) = f(x)g(y)$ is often denoted $f \otimes g$.

Observe that $f \otimes g \neq g \otimes f$.

One can define $T_1 \otimes T_2 \in S'(\mathbb{R}^2)$ for any pair $T_1, T_2 \in S'(\mathbb{R})$ by their action on $\varphi = \varphi_1 \otimes \varphi_2 \in S(\mathbb{R}^2)$:

$$\langle T_1 \otimes T_2, \varphi_1 \otimes \varphi_2 \rangle = T_1(\varphi_1)\, T_2(\varphi_2) .$$

Problem: is it enough to define $T_1 \otimes T_2$ on the subset of $S(\mathbb{R}^2)$ given by $\varphi = \varphi_1 \otimes \varphi_2$?

Solution One can approximate any $\varphi \in S(\mathbb{R}^2)$ by a sum of tensor products:

$$\varphi(x,y) = \sum_{k=1}^{N} \varphi_{1k}(x)\,\varphi_{2k}(y) + O(\varepsilon),$$

which means that $\varphi_N(x_1,x_2) = \sum_{k=1}^{N} \varphi_{1k}(x)\varphi_{2k}(y) \to \varphi(x,y)$ in $S(\mathbb{R}^2)$ when $N \to \infty$.

Then $\langle T_1 \otimes T_2, \varphi \rangle = \lim_{N\to\infty} \langle T_1 \otimes T_2, \sum_{k=1}^{N} \varphi_{1k} \otimes \varphi_{2k} \rangle$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} T_1(\varphi_{1k})\, T_2(\varphi_{2k})$$

The Fourier transform of tensor products

Let $\varphi = \varphi_1 \otimes \varphi_2$, i.e., $\varphi(x,y) = \varphi_1(x)\,\varphi_2(y)$.

Then $\hat{\varphi}(\xi,\eta) = \hat{\varphi}_1(\xi)\,\hat{\varphi}_2(\eta)$ (check that!).

This makes it easy to compute the Fourier transform of many $\varphi : \mathbb{R}^2 \to \mathbb{C}$ and tempered distributions $T \in S'(\mathbb{R}^2)$.

Ex. $$\frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2}} \varphi_1 \otimes \varphi_2 = \left(\frac{\partial^{k_1}}{\partial x_1} \varphi_1\right) \otimes \left(\frac{\partial^{k_2}}{\partial x_2} \varphi_2\right)$$

$$\Rightarrow (2\pi i \xi_1)^{k_1} \hat{\varphi}_1(\xi_1)\,(2\pi i \xi_2)^{k_2} \hat{\varphi}_2(\xi_2).$$

The Hankel transform and relatives

Many variations of the Fourier transform are derived from the usual Fourier transform in $\mathbb{R}^n$, by using certain symmetries. The Hankel transform is one example.

Suppose that $\varphi(x,y) = \psi(r)$ where $r = \sqrt{x^2+y^2}$: $\varphi(x,y)$ is invariant under rotations in the plane $\mathbb{R}^2$, or in other words, rotationally symmetric.

Then

$$\hat{\varphi}(\xi,\eta) = \iint_{\mathbb{R}^2} e^{-2\pi i(x\xi + y\eta)} \varphi(x,y)\,dx\,dy = \begin{pmatrix}\text{use polar} \\ \text{coordinates}\end{pmatrix}$$

$$= \int_0^\infty \int_0^{2\pi} e^{-2\pi i r(\xi\cos\theta + \eta\sin\theta)} \psi(r)\, r\,dr\,d\theta.$$

$$= \int_0^\infty \int_0^{2\pi} e^{-2\pi i r(\xi\cos\theta+\eta\sin\theta)}\,d\theta\; \psi(r)\, r\,dr.$$

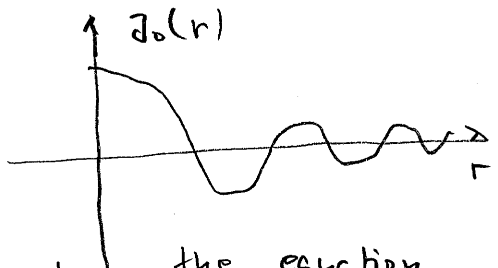
But $\xi\cos\theta + \eta\sin\theta = \sqrt{\xi^2+\eta^2}\cos(\theta+\lambda)$ for some λ.

Let $\rho = \sqrt{\xi^2+\eta^2}$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-2\pi i r(\xi\cos\theta + \eta\sin\theta)}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-2\pi i r\rho\cos(\theta+\lambda)}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-2\pi i r\rho\cos\theta}\,d\theta := J_0(2\pi r\rho)$$

This is the Bessel function of order 0.



It solves the equation

$$x^2 f''(x) + x f'(x) + x^2 f(x) = 0.$$

Therefore $\hat{\varphi}(\xi,\eta) = \int_0^\infty 2\pi J_0(2\pi r\rho)\,\psi(r)\, r\,dr = \tilde{\psi}(\rho)$.

<u>Def</u> $F(q) = 2\pi \int_0^\infty J_0(2\pi q r) f(r)\, r\, dr$ is the (zeroth order) Hankel transform of f.

<u>Note</u> The inverse transform is

$$f(r) = 2\pi \int_0^\infty F(q)\, J_0(2\pi q r)\, q\, dq.$$

In general the Fourier transform of a rotationally symmetric function $f: \mathbb{R}^n \to \mathbb{C}$ is computed in a similar way: if $f(x_1,\dots,x_n) = g(\sqrt{x_1^2+\dots+x_n^2}) = g(r)$,

$$\hat{f}(\xi_1,\dots,\xi_n) = \int_{\mathbb{R}^n} e^{-2\pi i (x_1\xi_1+\dots+x_n\xi_n)} g(r)\, dx_1\dots dx_n.$$

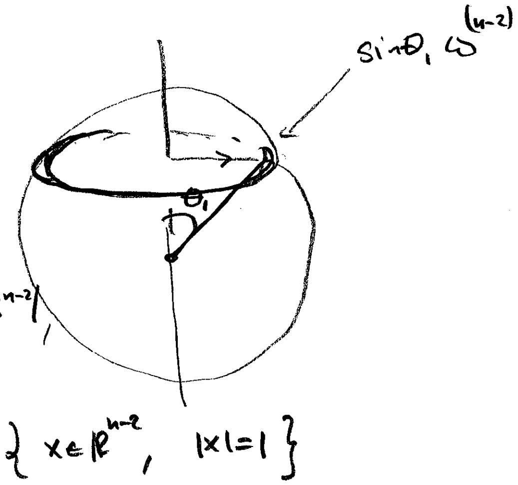
In polar coordinates,

$(x_1,\dots,x_n) = r(\cos\theta_1, \sin\theta_1\cos\theta_2, \dots\dots) \equiv r\omega$ where $|\omega|=1$.

Then $\hat{f}(\xi_1,\dots,\xi_n) = \int_0^\infty \int_{S^{n-1}} e^{-2\pi i \rho r \Omega\cdot\omega}\, d\omega^{n-1}\, g(r)\, r^{n-1} dr$

Here $\int_{S^{n-1}} e^{-2\pi i z \Omega\cdot\omega} d\omega^{n-1}$ only depends on the scalar product between Ω and ω, and therefore we may choose coordinates on S^{n-1} so that $\omega = (\omega_1,\dots,\omega_n)$ and $\Omega\cdot\omega = \omega_1$.

Also, $d\omega^{n-1} = d\theta_1 (\sin\theta_1)^{n-2} d\omega^{n-2}$



$$\Rightarrow \int_{S^{n-1}} e^{-2\pi i z \Omega\cdot\omega} d\omega^{n-1} = \int_0^\pi e^{-2\pi i \cos\theta\, z} (\sin\theta)^{n-2} d\theta \cdot |S^{n-2}|$$

where $|S^{n-2}|$ is the "area" of $S^{n-2} = \{x\in\mathbb{R}^{n-1}, |x|=1\}$

Homogeneous functions and distributions

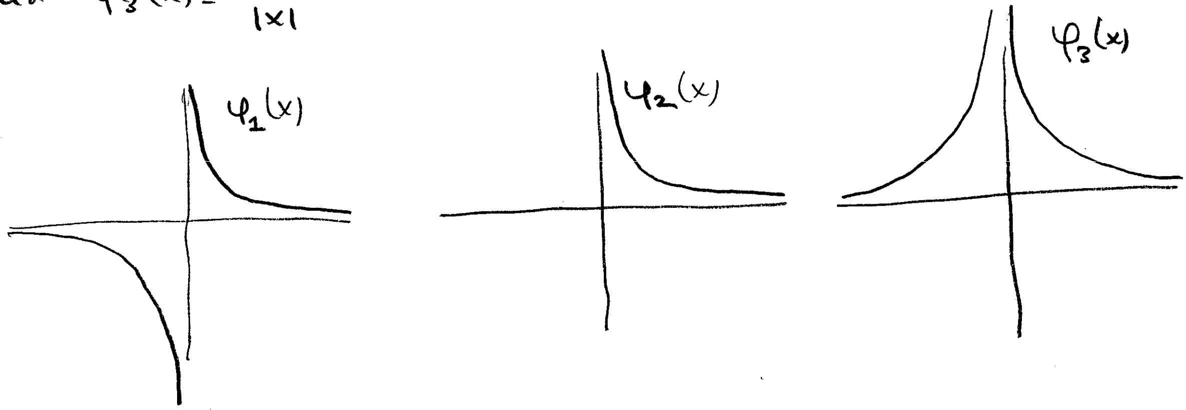
Def A function $f: \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called homogeneous of degree k if

$$f(tx) = t^k f(x) \quad \text{for all } x \neq 0.$$

Example $\varphi_1(x) = \frac{1}{x}$ is homogeneous of degree -1.

But that is true also for $\varphi_2(x) = \begin{cases} 1/x & \text{when } x > 0 \\ 0 & \text{otherwise} \end{cases}$

and $\varphi_3(x) = \frac{1}{|x|}$



Note that if f is homogeneous of degree k,

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx \underset{\substack{\uparrow \\ \text{from the} \\ \text{homogeneity}}}{=} \int_{\mathbb{R}^n} f(\alpha x)\,\alpha^{-k}\,\varphi(x)\,dx =$$

$$= \int_{\mathbb{R}^n} f(y)\,\alpha^{-k}\varphi\left(\frac{y}{\alpha}\right)\frac{dy}{\alpha^n} = \left\langle f, \frac{1}{\alpha^{n+k}}\varphi\left(\frac{\cdot}{\alpha}\right)\right\rangle$$

A distribution $T \in S'(\mathbb{R}^n)$ is homogeneous of degree k if

$$\langle T, \varphi \rangle = \left\langle T, \frac{1}{\alpha^{n+k}}\varphi\left(\frac{\cdot}{\alpha}\right)\right\rangle \quad \text{for all } \varphi \in S(\mathbb{R}^n).$$

Ex $\langle \delta, \varphi \rangle = \varphi(0) = \varphi\left(\frac{0}{\alpha}\right)\frac{1}{\alpha^{k+n}}$

$\Rightarrow \delta$ is homogeneous of degree $-n$.

The Fourier transform of homogeneous functions and distributions

Formally, if $f:\mathbb{R}\to\mathbb{R}$ is homogeneous of degree k, and

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)dx \,, \quad \text{then}$$

$$\hat{f}(t\xi) = \int_{\mathbb{R}} e^{-2\pi i t x\xi} f(x)\,dx \underset{\substack{\uparrow \\ \text{by a change of variables}}}{=} \int_{\mathbb{R}} e^{-2\pi i y\xi} f\left(\frac{y}{t}\right)\frac{dy}{t} \underset{\substack{\uparrow \\ \text{by the homogeneity of } f}}{=}$$

$$= \int_{\mathbb{R}} e^{-2\pi i y \xi} f(y)\frac{dy}{t^{k+1}} = \frac{1}{t^{k+1}}\hat{f}(\xi)$$

So, if f is homogeneous of degree k, then $\hat{f}$ is homogeneous of degree $-k-1$. But note that this may not be well defined!

However if $T\in S'(\mathbb{R})$ is homogeneous of degree λ, then

$$\langle \hat{T},\varphi\rangle = \langle T,\hat{\varphi}\rangle = \langle T, \frac{1}{t^{1+\lambda}}\hat{\varphi}\left(\frac{\cdot}{t}\right)\rangle, \quad \text{and}$$

because

$$\frac{1}{t^{1+\lambda}}\hat{\varphi}\left(\frac{\xi}{t}\right) = \frac{1}{t^{1+\lambda}}\int_{\mathbb{R}} e^{-2\pi i x\frac{\xi}{t}}\varphi(x)\,dx = \int_{\mathbb{R}} e^{-2\pi i \xi y}\varphi(ty)\frac{1}{t^{\lambda}}\,dy$$

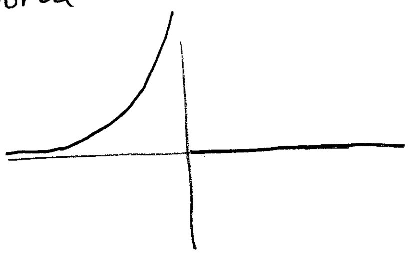
$$\Rightarrow \quad \langle \hat{T},\varphi\rangle = \langle T, \mathcal{F}(\varphi(t\cdot))\frac{1}{t^{\lambda}}\rangle = \langle \hat{T}, t^{-\lambda}\varphi(t\cdot)\rangle$$

and therefore $\hat{T}$ is homogeneous of degree $-\lambda-1$.

(recall $\langle \hat{T},\varphi\rangle = \langle \hat{T}, t^{\alpha+1}\varphi(t\cdot)\rangle$ so we need to set $\alpha=-\lambda-1$).

<u>Example</u> Compute the Fourier transform of

$$f(x) = \begin{cases} (-x)^{-1/2} & (x<0) \\ 0 & \text{otherwise} \end{cases}$$



<u>Solution:</u>

Let $f(x) = \frac{1}{2}\left(\frac{1}{|x|^{1/2}} - \operatorname{sgn} x \frac{1}{|x|^{1/2}} \right)$

$f_1(x) = |x|^{-1/2}$ is even, real and homogeneous of degree $-1/2$

$\Rightarrow \hat{f}_1(\xi)$ is even, real and homogeneous of degree $-(-\frac{1}{2}) - 1 = -\frac{1}{2}$.

Therefore $\hat{f}_1(\xi) = \pm c|\xi|^{-1/2}$.

In the usual way, we have

$$\langle \hat{f}_1, e^{-\pi \cdot^2} \rangle = \langle f_1, e^{-\pi \cdot^2} \rangle > 0 \quad \Rightarrow \quad \hat{f}_1(\xi) = |\xi|^{-1/2}.$$

$f_2(x) = \operatorname{sgn}(x) \frac{1}{|x|^{1/2}}$ is odd, real and homogeneous of degree $-1/2$, so

$\hat{f}_2(\xi)$ is odd, imaginary and homogeneous of degree $-1/2$.

Therefore $\hat{f}_2(\xi) = i c \operatorname{sgn} \xi \, |\xi|^{-1/2}$.

But what is $c \in \mathbb{R}$?

We have $2\pi i \xi \hat{f}_2(\xi) = \mathcal{F}(f_2')$, so

$$\langle 2\pi i (\cdot) \hat{f}_2, \varphi \rangle = \langle \mathcal{F}(f_2'), \varphi \rangle = \langle f_2', \hat{\varphi} \rangle = -\langle f_2, \hat{\varphi}' \rangle.$$

Then

$$\langle -2\pi c |\cdot|^{1/2}, e^{-\pi \cdot^2} \rangle = -\langle \operatorname{sgn}(\cdot)|\cdot|^{-1/2}, 2\pi(\cdot) e^{-\pi \cdot^2} \rangle$$

$$= -2\pi \langle |\cdot|^{1/2}, e^{-\pi \cdot^2} \rangle$$

so we must have $c = 1$.

Finally

$$\hat{f}(\xi) = \frac{1}{2}\left(\hat{f}_1(\xi) - \hat{f}_2(\xi) \right) =$$
$$= \frac{1}{2}\left(|\xi|^{-1/2} - i \operatorname{sgn} \xi \, |\xi|^{-1/2} \right) \qquad \text{for } \xi \in \mathbb{R}.$$

For $\xi \in \mathbb{R}$, $\xi > 0$,

$$\hat{f}(\xi) = \frac{1-i}{2} \, \xi^{-1/2}.$$ This can be extended to an analytical function

$$\hat{f}(\xi) = \frac{1}{(2i\xi)^{1/2}} \quad \text{in } \xi \in \mathbb{C} \setminus \{ z \in \mathbb{C} : \operatorname{Im} z \le 0, \operatorname{Re} z = 0 \}.$$

The Abel transform

Let
$$k(r,x) = \begin{cases} 2r\,(r^2 - x^2)^{-1/2} & (r > x) \\ 0 & \text{otherwise} \end{cases}$$

and let $f : [0,\infty) \to \mathbb{C}$.

Def The Abel transform of f is given by

$$f_A(x) = \int_0^\infty k(r,x)\, f(r)\, dr.$$

Inversion of the Abel transform

Let $\xi = x^2$ and $\rho = r^2$ and define $F_A(x^2) = f_A(x)$
$F(r^2) = f(r)$.

Then
$$F_A(\xi) = \int_0^\infty K(\xi - \rho)\, F(\rho)\, d\rho \;=\; K * F$$

where
$$K(z) = \begin{cases} \dfrac{1}{(-z)^{1/2}} & (z < 0) \\ 0 & \text{otherwise} \end{cases}$$

The map $F \longmapsto F_A$ is called the "modified Abel transform".

$$F_A(\xi) = K * F(\xi)$$

implies that

$$\mathcal{F}(F_A) = \mathcal{F}(K)\,\mathcal{F}(F),$$

and recall: $\mathcal{F}(K) = \dfrac{1}{(-2i\xi)^{1/2}}$. Then we may see that

$$\mathcal{F}(F) = (-2i\xi)^{1/2}\,\mathcal{F}(F_A)(\xi)$$

$$= -\frac{1}{\pi}\,\frac{1}{(-2i\xi)^{1/2}}\,2\pi i\xi\,\mathcal{F}(F_A)(\xi)$$

$$= -\frac{1}{\pi}\,\frac{1}{(-2i\xi)^{1/2}}\,\mathcal{F}(F_A') = -\frac{1}{\pi}\,\mathcal{F}(K)\,\mathcal{F}(F_A').$$

We then see that

$$F(\rho) = -\frac{1}{\pi}\,K * F_A'(\rho).$$

In the original variables

$$f(r) = -\frac{1}{\pi}\int_r^\infty \frac{f_A'(x)}{\sqrt{x^2-r^2}}\,dx$$

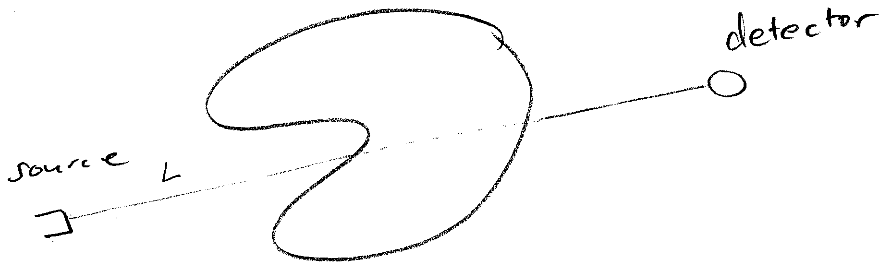
Note that

$$K * K * F' = -\pi F$$

so the operator

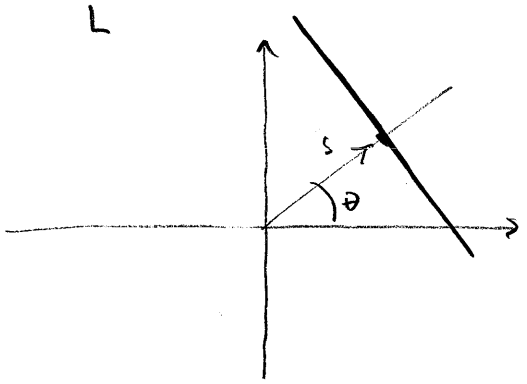
$F \longmapsto \dfrac{K}{\sqrt{\pi}} * F$ corresponds to taking a "half order integral" of F.

The Radon transform



Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the absorbtion per unit length of a ray passing through the object:

$$\int_L f(x(\ell))\, d\ell = \text{absorbtion along the line } L$$



we identify the angle θ and the unit vector θ.

Let $x = (x_1, x_2)$.

Def The Radon transform of $f \in S(\mathbb{R}^2)$ is

$$R_\theta f(s) = \int_{x\cdot\theta = s} f(x)\,dx = \int_{\mathbb{R}^2} f(x_1, x_2)\, \delta(s - x\cdot\theta)\, dx_1\, dx_2$$

Note that if f is a radial function, then

$$R_\theta f(s) = Af(s)$$

i.e. the Abel transform.

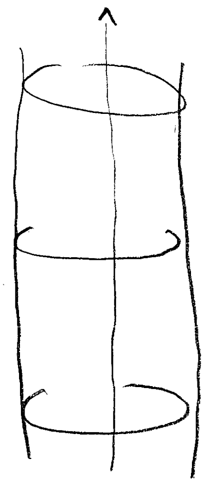
(this is an exercise to prove).

<u>Theorem</u> if $f \in S(\mathbb{R}^2)$ and $R_\theta f(s) = \int_{x\cdot\theta = s} f(x)\,dx$,

then $(R_\theta f)^\wedge(\sigma) = \hat{f}(\theta\sigma)$,

where $(R_\theta f)^\wedge(\sigma)$ is the usual Fourier transform with respect to s.

<u>Proof</u> Note first that the natural domain of definition for $R_\theta f(s)$ is $\mathbb{R} \times S^1$.
But from the definition it is also clear that $R_{-\theta} f(-s) = R_\theta f(s)$.



we compute $(R_\theta f)^\wedge(\sigma)$:

$$(R_\theta f)^\wedge(\sigma) = \int_{-\infty}^{\infty} e^{-2\pi i \sigma s} R_\theta f(s)\,ds$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i \sigma s} \int_{\mathbb{R}^2} f(x)\,\delta(x\cdot\theta - s)\,dx\,ds$$ (first compute the ds-integral)

$$= \int_{\mathbb{R}^2} f(x)\, e^{-2\pi i \sigma\, x\cdot\theta}\,dx = \hat{f}(\sigma\theta),$$ where

$\hat{f}(\xi)$ is the usual Fourier transform in $\mathbb{R}^2$.

<u>Corollary</u>: If $f \in S(\mathbb{R}^2)$ and $R_\theta f(s) \in S(\mathbb{R}\times S^1)$,
then f can be recovered from $R_\theta f(s)$ by first obtaining $\hat{f}(\xi)$ by the theorem above, and then computing the 2-dimensional inverse Fourier transform.

This is not efficient when discretizing in polar coordinates.

A better inversion formula for the Radon transform

Let $f \in S(\mathbb{R}^2)$. Then

$$f(x) = \int_{\mathbb{R}^2} e^{2\pi i x \xi} \hat{f}(\xi)\, d\xi =$$

$$= \int_0^\infty \int_{S^1} e^{2\pi i x \cdot \theta\sigma} \hat{f}(\sigma\theta)\, \underbrace{\sigma}_{\geq 1}\, d\theta\, d\sigma \qquad \text{(polar coordinates)}$$

$$= \frac{1}{2} \int_{S^1} \int_{\mathbb{R}} e^{2\pi i \sigma\theta \cdot x} \hat{f}(\sigma\theta)\, \underbrace{\sigma\, \mathrm{sgn}(\sigma)}_{=|\sigma|}\, d\sigma\, d\theta$$

$$= \frac{1}{2} \int_{S^1} \int_{\mathbb{R}} e^{2\pi i \theta x} (R_\theta f)^\wedge(\sigma)\, \sigma\, \mathrm{sgn}\, \sigma\, d\sigma\, d\theta. \qquad \circledast$$

Recall that

$$2\pi i \sigma \hat{f}(\sigma) = \mathcal{F}(f')(\sigma)$$

$$\mathrm{sign}\, \sigma = \mathcal{F}\left(\frac{i}{\pi(\cdot)}\right) \qquad \longleftarrow \text{[illegible] } 2\pi \ldots$$

Therefore

$$(R_\theta f)^\wedge(\sigma)\, \sigma\, \mathrm{sgn}\,\sigma = \frac{1}{2\pi i} \mathcal{F}\left(\frac{d}{ds} R_\theta f(\cdot) * \frac{i}{\pi(\cdot)}\right)(\sigma)$$

$$\Rightarrow \circledast = \frac{1}{4\pi^2} \int_{S^1} \underbrace{\left((R_\theta f)' * \frac{1}{(\cdot)}\right)(\theta x)}\, d\theta$$

$$= \int_{\mathbb{R}} \frac{(R_\theta f)'(u)}{u - s}\, du$$

<u>Note</u> The map

$$S(\mathbb{R} \times S^1) \longrightarrow S(\mathbb{R})$$

$$g(s, \theta) \longmapsto \int_{S^1} g(x \cdot \theta, \theta)\, d\theta$$

is "the dual" of the Radon transform, and it is denoted R^*

The reason for the name "dual" is the following.

Take $\varphi \in S(\mathbb{R}\times S^1)$. We define

$$\langle g, \varphi\rangle = \int_{S^1}\int_{\mathbb{R}} g(s,\theta)\,\varphi(s,\theta)\,ds\,d\theta .$$

Then

$$\langle Rf, \varphi\rangle = \int_{S^1}\int_{\mathbb{R}} R_\theta f(s)\,\varphi(s,\theta)\,ds\,d\theta$$

$$= \int_{S^1}\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(x)\,\delta(x\cdot\theta - s)\,\varphi(s,\theta)\,dx\,ds\,d\theta$$

$$= \int_{\mathbb{R}^2} f(x) \int_{S^1}\int_{\mathbb{R}} \delta(x\cdot\theta - s)\,\varphi(s,\theta)\,ds\,d\theta\,dx$$

$$= \int_{\mathbb{R}^2} f(x) \underbrace{\int_{S^1} \varphi(x\cdot\theta,\theta)\,d\theta}_{\in S(\mathbb{R}^2)}\,dx = \langle f, R^*\varphi\rangle$$

this is the usual $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^2$.

The Hilbert transform

Def $$F_{H_i}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{y-x}\,dx = \left(\frac{1}{\pi(-\cdot)} * f\right)(x)$$

Recall that $$\mathcal{F}\left(\frac{1}{\pi(-\cdot)}\right)(s) = i \operatorname{sgn} s$$

Hence

$$\left(\frac{1}{\pi(-\cdot)} * \frac{1}{\pi(-\cdot)} * f\right)^\wedge(s) = -(\operatorname{sgn} s)^2\hat{f} = -\hat{f}$$

and then

$$\frac{1}{\pi(-\cdot)} * \frac{1}{\pi(-\cdot)} * f = -f.$$ Therefore $f(x) = -\frac{1}{\pi(-\cdot)} * F_{H_i}(x)$

The analytical signal

Let $f \in S(\mathbb{R})$ be real valued and let

$$g(x) = f(x) - i\, F_{H_i}(x) .$$

This is called "the analytical signal". The Fourier transform of

$$\hat{g}(s) = \hat{f}(s) - i \; i \operatorname{sgn} s \; \hat{f}(s)$$

$$= \hat{f}(s) + \operatorname{sgn} s \; \hat{f}(s) .$$

Then $\hat{g}(s) = 0$ for $s \leq 0$.

Why is this called "the analytical signal"?

Let $\varphi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} f(\xi)\, d\xi$

Then $\begin{cases} \Delta \varphi = 0 & \text{in } y > 0 \\ \varphi(x,0) = f \end{cases}$

There is an analytic function $\Phi(z)$ so that

$\varphi(x,y) = \operatorname{Re} \Phi(z) \qquad (z = x + iy)$

Φ is analytic in $\operatorname{Im} z > 0$ and

$$\lim_{y \to 0^+} \operatorname{Im} \Phi(x+iy) = F_{H_i}(x) .$$