Proposition

The Plancherel formula : Let $f, \varphi \in S$.

Then $\int_{\mathbb{R}} \hat{f}(x)\varphi(x)\,dx = \int_{\mathbb{R}} f(x)\hat{\varphi}(x)\,dx$

Proof $\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi i xy} f(y)\,dy\ \varphi(x)\,dx =$

$= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i xy} f(y)\varphi(x)\,dx\,dy = \int_{\mathbb{R}} f(y) \int e^{-2\pi i xy}\varphi(x)\,dx\,dy$

The changes of order of integration are justified because $f, \varphi \in S$.

The Fourier transform of distributions

Because $\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$ for $f \in S$, we define the Fourier transform of $T \in S'$ by

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle \quad \text{for all } \varphi \in S.$$

Proposition $\hat{T} \in S'$ (this we have to prove!)

1) Linear : this is clear because $\varphi \mapsto \hat{\varphi}$ is linear.

2) Take $\varphi_n \to 0$ in S. Then $\hat{\varphi}_n \to 0$ in S

so $\langle \hat{T}, \varphi_n \rangle = \langle T, \hat{\varphi}_n \rangle \to 0$ when $n \to \infty$.

[note: that $\hat{\varphi}_n \to 0$ in S follows from the calculations needed to prove that $\hat{\varphi}_n \in S$]

Example Let $\beta \in \mathbb{Z}^+$. Compute $\mathcal{F}(D^\beta \delta_0)$

$$\langle \mathcal{F}(D^\beta \delta_0), \varphi \rangle = \langle D^\beta \delta, \hat{\varphi} \rangle =$$

$$= (-1)^\beta \langle \delta_0, D^\beta \hat{\varphi} \rangle = \langle \delta_0, \mathcal{F}((2\pi i \cdot)^\beta \varphi) \rangle$$

$$= \int e^{-i 2\pi \xi x} (2\pi i x)^\beta \varphi(x)\, dx \Big|_{\xi = 0}$$

$$= \int_{\mathbb{R}} (2\pi i x)^\beta \varphi(x)\, dx .$$

Conclusion: $\mathcal{F}(D^\beta \delta) = (2\pi i \cdot)^\beta$

Theorem the Fourier inversion formula for tempered distributions:

Let $\check{\varphi}(x) = \varphi(-x)$ when $\varphi \in \mathcal{S}$, and let $\check{T}$ be defined by $\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle$ for $T \in \mathcal{S}'$.

Then for all $T \in \mathcal{S}'$, $\mathcal{F}\mathcal{F} T = \check{T}$.

Proof $\check{\varphi}(x) = \varphi(-x) = \int_{\mathbb{R}} e^{2\pi i \xi(-x)} \hat{f}(\xi) d\xi = \mathcal{F}\hat{\varphi}(x) = \mathcal{F}\mathcal{F}\varphi(x)$.

Hence

$$\langle \mathcal{F}\mathcal{F} T, \varphi \rangle = \langle \mathcal{F} T, \mathcal{F}\varphi \rangle = \langle T, \mathcal{F}\mathcal{F}\varphi \rangle$$

$$= \langle T, \check{\varphi} \rangle = \langle \check{T}, \varphi \rangle$$

Further properties of the Fourier transform of tempered distributions

Proposition

1) The Fourier transform is linear:

$$\mathcal{F}(T_1 + T_2) = \mathcal{F}(T_1) + \mathcal{F}(T_2)$$
$$\mathcal{F}(cT_1) = c\,\mathcal{F}(T_1)$$

2) Let $T \in \mathcal{S}'$ and $f \in C^\infty$ and such that for all $\beta > 0$ there is α such that

$$\sup_{x \in \mathbb{R}} \left((1+x^2)^{-\alpha} \, |D^\beta f(x)| \right) < \infty$$

Then

- $\mathcal{F}(DT) = 2\pi i (\cdot)\, \mathcal{F}(T)$
- $\mathcal{F}(-2\pi i (\cdot) T) = D\hat{T}$
- $\mathcal{F}(fT) = \hat{f} * \hat{T}$ (this a <u>definition</u>)
- $\mathcal{F}(\hat{f} * T) = \check{f}\, \mathcal{F}(T)$
- $\mathcal{F}(\tau_s T) = e^{-2\pi i s(\cdot)}\, \hat{T}$
- $\mathcal{F}(e^{2\pi i (\cdot) s} T) = \tau_s \hat{T}$

and moreover

$$D(\hat{f} * T) = (D\hat{f}) * T + \hat{f} * DT$$

<u>Proof</u> All follows directly from the definitions, except that we have not yet defined the convolution of distributions.

Recall that if $\varphi_1, \varphi_2 \in S$, then

$$\mathcal{F}(\varphi_1 * \varphi_2) = \hat{\varphi}_1 \hat{\varphi}_2$$

If $\hat{T} \in S'$ and $\hat{f} \in S'$ and $f \in C^\infty$ and has moderate growth, then $fT \in S'$ (because we can multiply distributions and functions).

Hence we can <u>define</u> $\hat{f} * \hat{T}$ by

$$\hat{f} * \hat{T} = \mathcal{F}(fT), \quad \text{i.e.}$$

$$\langle \hat{f} * \hat{T}, \varphi \rangle = \langle \mathcal{F}(fT), \varphi \rangle = \langle fT, \hat{\varphi} \rangle .$$

Taking Fourier transforms, we find

$$\check{f}\, \check{T} = \mathcal{F}(\hat{f} * \hat{T}) \iff \check{f}\, \hat{T} = \mathcal{F}(\hat{f} * T)$$

Also

$$\mathcal{F}(D(\hat{f} * T)) = \underbrace{(2\pi i(\cdot))\, \check{f}}_{= \mathcal{F}(D\hat{f})}\, \hat{T} = \mathcal{F}(D\hat{f} * T) = \mathcal{F}(\hat{f} * DT) .$$

<u>Differential equations</u>

Recall that if $f \in C^1(\mathbb{R})$ and $f' \equiv 0$, then f is a constant.

What can we say if $T \in S'(\mathbb{R})$ and $DT = 0$ in $S'(\mathbb{R})$?

__Lemma__ Let $T \in S'$, and assume that

$(\cdot) T = 0$ in S'. Then there is $a \in \mathbb{C}$ such that $T = a\delta$

__Proof__ Let $\psi \in S$ with $\psi(0) = 0$, and let $\varphi(x) = \psi(x)/x$. Then $\varphi \in S(\mathbb{R})$..

For $T \in S'(\mathbb{R})$

$$T(\psi) = T((\cdot)\varphi) = \underbrace{(\cdot) T(\varphi)}_{} = 0$$

note: this should be interpreted as

$$[(\cdot)T](\varphi) = \langle (\cdot)T, \varphi \rangle.$$

Next we take $\varphi \in S$ arbitrarily, and fix $\varphi_1 \in S$ with $\varphi_1(0) = 1$. We can write

$$\varphi(x) = \underbrace{\varphi(x) - \varphi(0)\varphi_1(x)}_{= \psi(x)} + \varphi(0)\varphi_1(x)$$

Then

$$T(\varphi) = T(\psi) + T(\varphi(0)\varphi_1) = T(\varphi_1)\varphi(0)$$

so $T = T(\varphi_1)\delta$.

Note that the result depends on the choice of φ_1, but in the C^1-case, $f' \equiv 0$ implies that f is a constant but it does not say which constant.

Corollary Let $T \in S'(\mathbb{R})$ and assume that $DT = 0$. Then $\hat{T} = a\delta$ for some $a \in \mathbb{C}$

Proof

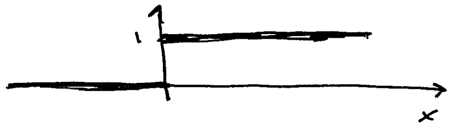
$$0 = \mathcal{F}(DT) = (2\pi i(\cdot))\,\hat{T}$$

$$\Rightarrow \quad \hat{T} = a\delta \quad \text{for some } a \in \mathbb{C}$$

$$\Rightarrow \quad T = a, \quad a \text{ constant}.$$

Example Let H be the Heavyside step function,

$$H(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x < 0 \end{cases}$$

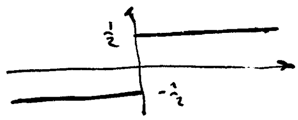


Then $\mathcal{F}H = \dfrac{1}{2\pi i(\cdot)} + \dfrac{1}{2}\delta$, where

$$\left\langle \frac{1}{\cdot}, \varphi \right\rangle = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{1}{2\pi i x}\, \varphi(x)\, dx$$

(the Cauchy principal value)

Proof Let $H_1 = H - \frac{1}{2}$



Then $DH_1 = DH = \delta$, and hence

$$2\pi i(\cdot)\,\widehat{H_1} = 1 \quad \text{and consequently}$$

$$\widehat{H_1} = \frac{1}{2\pi i\,\cdot} + a\delta$$

(why do you have to add $a\delta$?)

But H_1 is odd $\Rightarrow$ $\mathcal{F}H_1$ is odd.

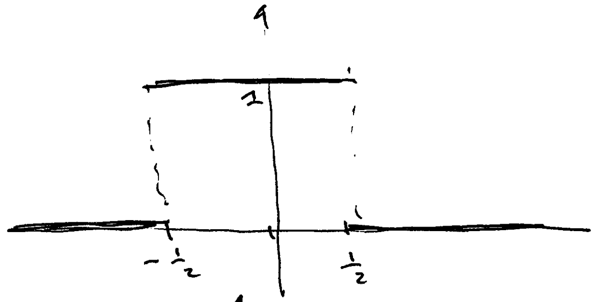
And $\frac{1}{(\cdot)}$ is odd, δ is even, and therefore $a = 0$.

Then $\hat{H} = \mathcal{F}(H_1 + \frac{1}{2}) = \dfrac{1}{2\pi i(\cdot)} + \dfrac{1}{2}\delta$

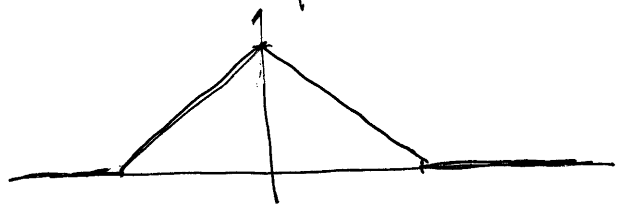
qed.

Common notation

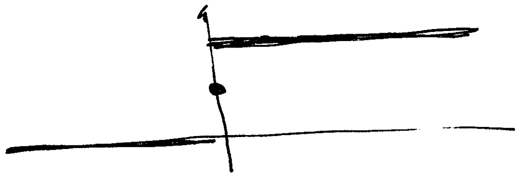
$$\Pi(x) = \begin{cases} 1 & \text{when } |x| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



$$\Lambda(x) = \begin{cases} 1-x & \text{when } 0 \le x < 1 \\ x+1 & \text{when } -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$$



$$H(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x < 0 \end{cases}$$



$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases} = 2H(x) - 1$$

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$$

<u>Note</u> Check that $\int_{-\infty}^{\infty} \operatorname{sinc} x \, dx = 1$, if the divergent integral is properly defined.

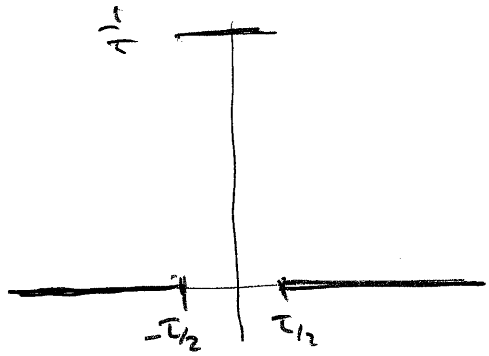
$$\mathcal{F}(\Pi) = \int_{-1/2}^{1/2} e^{-2\pi i x \xi} \, dx = \frac{e^{\pi i \xi} - e^{-\pi i \xi}}{2\pi i \xi} = \frac{\sin \pi \xi}{\pi \xi} = \operatorname{sinc} \xi .$$

Both Π and sinc are even, so

$$\Pi = \mathcal{F}(\operatorname{sinc}(\cdot)), \text{ and}$$

$$\Pi(0) = \int_{-\infty}^{\infty} \mathcal{F}(\Pi) \, d\xi = \int_{-\infty}^{\infty} \frac{\sin \pi \xi}{\pi \xi} \, d\xi .$$

Distributions and generalized functions



Let $\Pi(x) = \begin{cases} 1 & \text{when } |x| < \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$

Then $\frac{1}{\tau}\Pi\left(\frac{x}{\tau}\right) = \begin{cases} 1/\tau & \text{when } |x| < \frac{\tau}{2} \\ 0 & \text{elsewhere} \end{cases}$

Then $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\tau}\Pi\left(\frac{x}{\tau}\right) dx = \int_{-\infty}^{\infty} \Pi(x)\, dx = 1$

and for $\varphi \in C(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \frac{1}{\tau}\Pi\left(\frac{x}{\tau}\right)\varphi(x)\, dx = \int_{-\infty}^{\infty} \Pi(x)\,\varphi(\tau x)\, dx = \int_{-1/2}^{1/2} \varphi(\tau x)\, dx$$

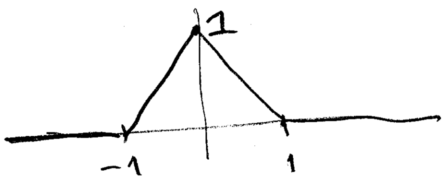
$\to \varphi(0)$ when $\tau \to 0$,

because $\varphi(\tau x) \to \varphi(0)$ uniformly in $-\frac{1}{2} \le x \le \frac{1}{2}$.

Hence we can see δ as the limit of $\frac{1}{\tau}\Pi\left(\frac{\cdot}{\tau}\right)$ when $\tau \to 0$

Def Let T_n be a family of distributions, $T_n \in S'(\mathbb{R})$. We say that $T_n \to T$ in S if for every $\varphi \in S(\mathbb{R})$,

$$\lim_{n \to 0} \langle T_n, \varphi \rangle = \langle T, \varphi \rangle.$$

Let $\Lambda(x) =$ 

Then, as before,

$$\int_{-\infty}^{\infty} \frac{1}{\tau}\Lambda\left(\frac{x}{\tau}\right)\varphi(x)\, dx = \int_{-\infty}^{\infty} \Lambda(x)\,\varphi(\tau x)\, dx \to \varphi(0)$$

when $\tau \to 0$

There are many other examples:

$$\int_{-\infty}^{\infty} \frac{1}{\tau} e^{-\pi \left(\frac{x}{\tau}\right)^2} \varphi(x)\,dx \to \varphi(0)$$

$$\int_{-\infty}^{\infty} \frac{1}{\tau} \operatorname{sinc}\left(\frac{x}{\tau}\right) \varphi(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\tau} \frac{\sin \pi x/\tau}{\pi x/\tau} \varphi(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} \varphi(\tau x)\,dx .$$

The last one is more difficult to prove, because

sinc x is <u>not</u> absolutely integrable, but

rather it is necessary to rely on cancellations due to

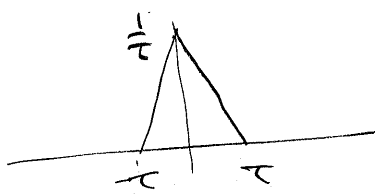
the oscillatory behaviour of $\sin \pi x$.

<u>What about derivatives of generalized functions?</u>

with $\delta = \lim_{\tau \to 0} \frac{1}{\tau} \Pi\left(\frac{\cdot}{\tau}\right)$ in S', you can't

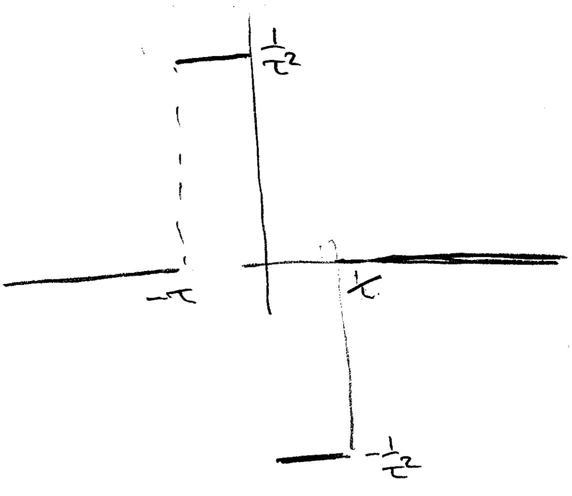
do much, but for

$\delta = \lim_{\tau \to 0} \frac{1}{\tau} \Lambda\left(\frac{\cdot}{\tau}\right)$



you have

$$D\left(\frac{1}{\tau} \Lambda\left(\frac{x}{\tau}\right)\right) = \frac{1}{\tau^2}\left(\Pi\left(\frac{x+\tau/2}{\tau}\right) - \Pi\left(\frac{x-\tau/2}{\tau}\right)\right)$$



Hence

$$\int_{-\infty}^{\infty} D\left(\frac{1}{\tau}\Lambda\left(\frac{x}{\tau}\right)\right)\varphi(x)\,dx =$$

$$= \int_{-\infty}^{\infty}\left(\Pi\left(\frac{x+\tau/2}{\tau}\right)-\Pi\left(\frac{x-\tau/2}{\tau}\right)\right)\frac{1}{\tau^2}\,\varphi(x)\,dx$$

$$= \int_{-\infty}^{\infty}\left(\Pi\left(x+\tfrac{1}{2}\right)-\Pi\left(x-\tfrac{1}{2}\right)\right)\frac{1}{\tau}\,\varphi(\tau x)\,dx$$

$$= \int_{-1/2}^{1/2}\frac{\varphi(\tau(x-\frac{1}{2}))-\varphi(\tau(x+\frac{1}{2}))}{\tau}\,dx \underset{\text{Taylor}}{=} \int_{-1/2}^{1/2}\frac{1}{\tau}\left(\varphi'(0)\left(\tau\left(x-\tfrac{1}{2}\right)-\tau\left(x+\tfrac{1}{2}\right)\right)+O(\tau^2)\right)dx$$

$$= \varphi'(0)\int_{-1/2}^{1/2}(-1+O(\tau))\,d\tau \;\to\; -\varphi'(0) \quad \text{when } \tau\to 0.$$

But we also have $\langle D\delta,\varphi\rangle = -\varphi'(0)$, so it seems that $D\left(\frac{1}{\tau}\Lambda\left(\frac{\cdot}{\tau}\right)\right) \to \delta'$ in S'.

Is this always true? Could you compute

$$\lim_{\tau\to 0} D\left(\frac{1}{\tau}\operatorname{sinc}\left(\frac{x}{\tau}\right)\right)\ ?$$

The great advantage with studying S and S' is that you move all difficult operations to $\varphi \in S$.

The Poisson summation formula

Let $\varphi \in \mathcal{S}$. Then $$\sum_{k\in\mathbb{Z}} \hat{\varphi}(k+\xi) = \sum_{k\in\mathbb{Z}} \varphi(k) e^{-2\pi i k\xi} \qquad (*)$$

Note The left hand side is periodic with period 1, and this is true also for the RHS, because $e^{-2\pi i k\xi}$ is periodic.

Proof

Recall that $$\mathcal{F}(e^{2\pi i (\cdot) s}\varphi) = \tau_s \hat{\varphi}$$

and that $$e^{-2\pi i s(\cdot)} \hat{T} = \mathcal{F}(\tau_s T).$$

Here $\varphi \in \mathcal{S}$ and $T \in \mathcal{S}'$.

Hence $$\sum_{k\in\mathbb{Z}} \hat{\varphi}(k+\xi) = \sum_{k\in\mathbb{Z}} \tau_{-\xi} \hat{\varphi}(k)$$

$$= \sum_{k\in\mathbb{Z}} \mathcal{F}(e^{-2\pi\xi(\cdot)} \hat{\varphi})(k) = \sum_{k\in\mathbb{Z}} \langle \delta_k, \mathcal{F}(e^{-2\pi i \xi(\cdot)}\varphi)\rangle$$

$$= \langle \sum_{k\in\mathbb{Z}} \delta_k, \mathcal{F}(e^{-2\pi i \xi(\cdot)}\varphi)\rangle = \sum_{k\in\mathbb{Z}} \varphi(k) e^{-2\pi i k\xi}$$

So the left hand side of $(*)$ is
$$\langle \mathcal{F}(\sum_k \delta_k), e^{-2\pi i k(\cdot)}\varphi\rangle$$ and

the right hand side is
$$\langle \sum_k \delta_k, e^{-2\pi i k(\cdot)}\varphi\rangle.$$

We need to prove that $\mathcal{F}(\sum_k \delta_k) = \sum_k \delta_k$,

i.e. that $\sum_k \delta_k$ is a fixed point of $\mathcal{F}$

But this follows from

1) $\tau_{\pm 1}\left(\sum_{k\in\mathbb{Z}} \delta_k\right) = \sum_{k\in\mathbb{Z}} \delta_k$ (this is a translation invariant distribution; for integer translates)

2) $e^{2\pi i(\cdot)} \sum_{k\in\mathbb{Z}} \delta_k = \sum_{k\in\mathbb{Z}} \delta_k$ (because the exponential is equal to one at all integer points)

Taking Fourier transforms gives

1') $\mathcal{F}\left(\tau_1\left(\sum_{k\in\mathbb{Z}} \delta_k\right)\right) = e^{-2\pi i(\cdot)} \mathcal{F}\left(\sum_{k\in\mathbb{Z}} \delta_k\right) = \mathcal{F}\left(\sum_k \delta_k\right)$

2') $\mathcal{F}\left(e^{2\pi i(\cdot)} \sum_{k\in\mathbb{Z}} \delta_k\right) = \tau_1\left(\mathcal{F}\left(\sum_k \delta_k\right)\right) = \mathcal{F}\left(\sum_k \delta_k\right)$

From 1') we find that

$$0 = \left(e^{-2\pi i(\cdot)} - 1\right) \mathcal{F}\left(\sum_k \delta_k\right) = \frac{e^{-2\pi i(\cdot)} - 1}{(\cdot)} \, (\cdot) \, \mathcal{F}\left(\sum_k \delta_k\right)$$

But $\frac{e^{-2\pi i \xi} - 1}{\xi} \to -2\pi i \neq 0$ when $\xi \to 0$, so

$$(\cdot)\, \mathcal{F}\left(\sum_k \delta_k\right) = 0 \Rightarrow \mathcal{F}\left(\sum_k \delta_k\right) = a\delta \text{ for some } a \in \mathbb{C}.$$

From 2') we have that $\tau_{\pm 1}\left(\mathcal{F}\left(\sum_k \delta_k\right)\right) = \mathcal{F}\left(\sum_k \delta_k\right)$, i.e. the expression can be translated by integers. But then $\mathcal{F}\left(\sum_k \delta_k\right) = a \sum_k \delta_k$.

What is the constant?

$\langle \mathcal{F}\left(\sum_k \delta_k\right), \varphi \rangle = a \langle \sum \delta_k, \varphi \rangle$ and

$= \langle \sum_k \delta_k, \mathcal{F}\varphi \rangle$. With $\varphi = e^{-\pi^2 x} \Rightarrow \varphi = \hat{\varphi}$,

we find that $a = 1$.

<u>qed</u>

Example

If $\varphi \in S$ and $\hat{\varphi}(\xi) = 0$ when $|\xi| > 1$,

then $\sum_k \varphi(k) = \int_{\mathbb{R}} \varphi(x)\,dx$,

because

$$\sum_k \varphi(k) = \langle \Sigma \delta_k, \varphi \rangle = \langle \mathcal{F}(\Sigma \delta_k), \varphi \rangle$$

$$= \langle \Sigma \delta_k, \hat{\varphi} \rangle = \langle \delta_0, \hat{\varphi} \rangle = \hat{\varphi}(0)$$

$$= \langle 1, \varphi \rangle = \int_{\mathbb{R}} \varphi(x)\,dx.$$

Sampling

Definition

$$\operatorname{sinc} x = \begin{cases} \frac{\sin \pi x}{\pi x} & \text{when } x \neq 0 \\ 1 & x = 0 \end{cases}$$

$$\Pi(x) = \mathbb{1}_{]-\frac{1}{2},\frac{1}{2}[} = \begin{cases} 1 & \text{when } |x| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Note: $\mathcal{F}(\Pi) = \operatorname{sinc}(\cdot)$

Theorem Let $f \in C^\infty$ be a function with moderate growth (i.e., $\forall \beta \in \mathbb{N}\ \exists \alpha : \sup_x |(1+|x|)^{-\alpha} D^\beta f(x)| < \infty$

Assume in addition that $\hat{f}(s) = 0$ for $|s| \geq \frac{1}{2}$

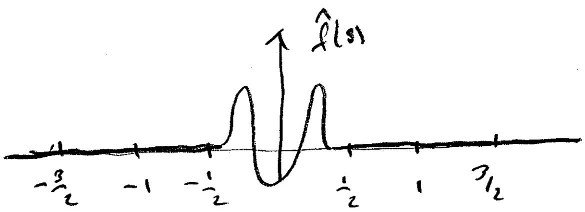
Then $$f = \operatorname{sinc} * \left(\sum_{k=-\infty}^{\infty} f(k)\delta_k \right)$$
$$= \sum_{k=-\infty}^{\infty} f(k) \operatorname{sinc}(\cdot - k)$$

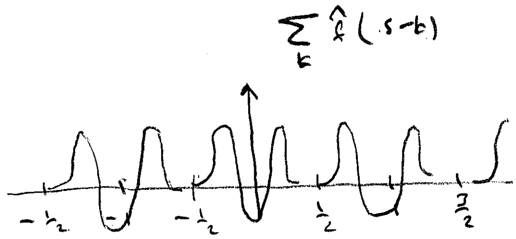
Proof Note that $\sum_{k=-\infty}^{\infty} f(k)\delta_k = f \sum_{k=-\infty}^{\infty} \delta_k$ in S'

By Fourier transform we then get

$$\mathcal{F}\left(\sum_k f(k)\delta_k\right) = \hat{f} * \mathcal{F}\left(\sum_k \delta_k\right) = \hat{f} * \sum \delta_k \qquad (*)$$
$$= \sum_k \hat{f}(\cdot - k)$$

The right hand side is periodic with period 1, because it is the sum of all translated copies of $\hat{f}$:





Note that because $\hat{f}(s) = 0$ outside $|s| \leq \frac{1}{2}$, the translated copies do not over lap, and hence

$$\hat{f}(s) = \left(\sum_k \hat{f}(s-k) \right) \Pi(s).$$

So then

$$\hat{f}(s) = \Big(\sum_k \hat{f}(s-k)\Big)\,\Pi(s) =$$

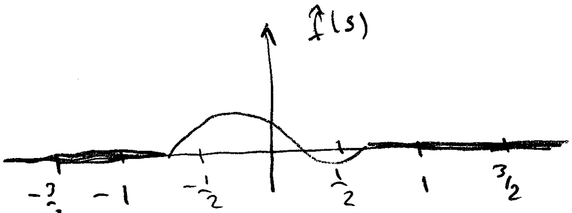
$$= \mathcal{F}\Big(\sum_k f(k)\delta_k\Big)(s)\cdot \mathcal{F}(\mathrm{sinc})(s)$$

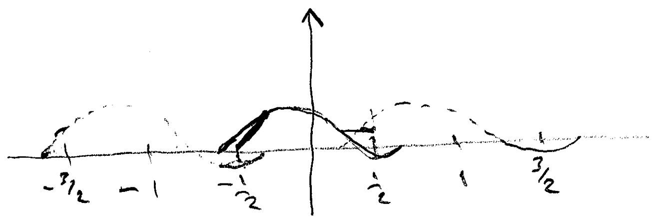
$$= \mathcal{F}\Big(\mathrm{sinc} * \sum_k f(k)\delta_k\Big)(s)$$

By inverse Fourier transform we get the result:

$$f = \sum_{k=-\infty}^{\infty} f(k)\,\mathrm{sinc}(\cdot - k)$$

<u>Aliasing</u> If $\hat{f}(s) \neq 0$ for $|s| \geq \frac{1}{2}$, the translated copies will overlap:





Then $\hat{f}(s) \neq \Big(\sum \hat{f}(s-k)\Big)\,\Pi(s)$.

This gives an error if one tries to recover a sampled signal by convolution with sinc.

It is called <u>aliasing</u>

The discrete Fourier transform

Prop

Let $\hat{T} \in S'$, and assume $\hat{T}_1 = \hat{T}$, i.e. $\hat{T}$ is periodic with period 1.

Assume that also T is periodic.

Then T has integer period, $T_N = T$,

and
$$T = \sum_k t_k \delta_k \qquad \hat{T} = \sum_k c_k \delta_{k/N}.$$

The two sequences t_k and c_k are periodic with period N, and are related by the discret Fourier transform:
$$N c_k = \sum_{l=1}^{N} t_l e^{-2\pi i k l/N} \qquad t_l = \sum_{k=1}^{N} c_k e^{2\pi i k l/N}$$

Note The discrete Fourier transform can be defined independently of T and $\hat{T}$.

Proof $\hat{T}_1 = \hat{T}$ implies that $T = \sum_{k=-\infty}^{\infty} t_k \delta_k$ for some $t_k \in \mathbb{C}$ (by a previous Lemma).

If T is periodic, $T_a = T$, a must be an integer, $a = N \in \mathbb{N}$. It follows that
$$\hat{T} = \sum_{k=-\infty}^{\infty} c_k \delta_{k/N}$$

[by the same argument as before:
$$T_a = T \Rightarrow e^{-2\pi i a \cdot} \hat{T} = \hat{T}$$
$$\Rightarrow (e^{-2\pi i a \cdot} - 1)\hat{T} = 0$$

and therefore $\hat{T}$ must be concentrated on the points k/a, $k \in \mathbb{Z}$]

Then $\mathcal{F}(T) = \mathcal{F}\left(\sum_k t_k \delta_k\right)$

$$= \mathcal{F}\left(\sum_{k=1}^{N} t_k \sum_{\ell=-\infty}^{\infty} \delta_{k+\ell N}\right)$$

(because the sequence $\{t_k\}$ is periodic)

$$= \sum_{k=1}^{N} t_k \, \mathcal{F}\left(\sum_{\ell=-\infty}^{\infty} \delta_{k+\ell N}\right)$$

$$= \sum_{k=1}^{N} t_k e^{-2\pi i k(\cdot)} \mathcal{F}\left(\sum_{\ell=-\infty}^{\infty} \delta_{\ell N}\right) = \sum_{k=1}^{N} t_k e^{-2\pi i k(\cdot)} \frac{1}{N} \sum_{\ell=-\infty}^{\infty} \delta_{\ell/N}$$

$$= \sum_{k=1}^{N} t_k e^{-2\pi i k \ell/N} \frac{1}{N} \sum_{\ell=-\infty}^{\infty} \delta_{\ell/N}$$

$$= \sum_{\ell=-\infty}^{\infty} \underbrace{\left(\frac{1}{N} \sum_{k=1}^{N} t_k e^{-2\pi i k\ell/N}\right)}_{c_k} \delta_{\ell/N}$$

It follows that $N c_\ell = \sum_{k=1}^{N} t_k e^{-2\pi i k\ell/N}$.

<u>Periodic functions</u> Assume that $f \in S$

Then $g(x) = \sum_{m=-\infty}^{\infty} f(x - m\tau)$ is periodic

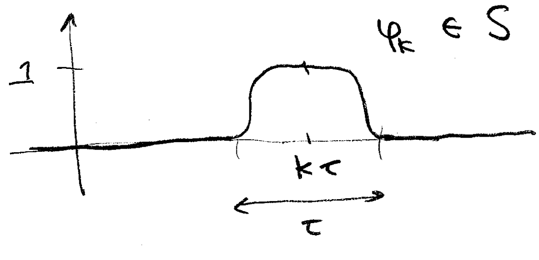
with period τ. As a distribution, $g \in S'$,

we find that

$$\hat{g} = \sum_{k=-\infty}^{\infty} c_k \delta_{k/\tau} \quad \text{and}$$

$$g = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i \frac{k}{\tau}(\cdot)}$$

$\varphi_k \in S$



We can compute the c_k by

$$c_k = \hat{g}(\varphi_k) = g(\hat{\varphi}_k) = \int_{-\infty}^{\infty} g(x)\, \hat{\varphi}_k(x)\, dx$$

i.e. $\langle \hat{g}, \varphi_k \rangle$

$$= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(x-m\tau)\, \hat{\varphi}_k(x)\, dx$$

$$= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-m\tau)\, \hat{\varphi}_k(x)\, dx = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, \hat{\varphi}_k(x+m\tau)\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \sum_{m=-\infty}^{\infty} \hat{\varphi}_k(x+m\tau)\, dx =$$

$$= \int_{-\infty}^{\infty} \tau f(\tau y) \sum_{m=-\infty}^{\infty} \hat{\varphi}_k(\tau(y+m))\, dy =$$

$$= \int_{-\infty}^{\infty} \tau f(\tau y) \sum_{m=-\infty}^{\infty} \mathcal{F}\left(\frac{1}{\tau}\varphi_k\left(\frac{\cdot}{\tau}\right)\right)(y+m)\, dy$$

$$= \int_{-\infty}^{\infty} f(\tau y) \sum_{m=-\infty}^{\infty} \underbrace{\varphi_k\left(\frac{m}{\tau}\right) e^{-2\pi i y m}}_{\substack{=0 \text{ when} \\ m \neq k}}\, dy$$

$$= \int_{-\infty}^{\infty} f(\tau y)\, e^{-2\pi i y k} = \frac{1}{\tau}\hat{f}\left(\frac{k}{\tau}\right)$$

So this is a method for calculating Fourier coefficients of certain periodic functions.