The Fourier transform: further properties

1) $f \supset \hat{f} \quad \Rightarrow \quad Df \supset$

$$-2\pi i(\cdot) f \supset D\hat{f}$$

$$\Big(\quad \hat{f}(\xi) = \int e^{-2\pi i x \xi} f(x)\, dx$$

$$D\hat{f}(\xi) = \int e^{-2\pi i x \xi}\left(-2\pi i x\, f(x)\right) dx = \mathcal{F}\left(-2\pi i(\cdot) f\right)$$

$$\mathcal{F}(Df) = \int e^{-2\pi i x \xi}\, Df(x)\, dx = -\int D\left(e^{-2\pi i x \xi}\right) f\, dx$$

$$= 2\pi i \xi \int e^{-2\pi i x \xi} f(x)\, dx \quad \Big)$$

2) Translation: $\tau_a : f(\cdot) \mapsto f(\cdot - a)$

$$\mathcal{F}(\tau_a f)(\xi) = \int e^{-2\pi i x \xi} f(\underbrace{x-a}_{y})\, dx =$$

$$= \int e^{-2\pi i (y+a)\xi} f(y)\, dy = e^{-2\pi i a \xi}\, \hat{f}(\xi)$$

$$\tau_a \hat{f}(\xi) = \int e^{-2\pi i x(\xi - a)} f(x)\, dx =$$

$$= \int e^{-2\pi i x \xi}\left[e^{2\pi i a x} f(x)\right] dx = \mathcal{F}\left(e^{2\pi i a(\cdot)} f\right)(\xi)$$

1a)	$-2\pi i(\cdot) f \supset D\hat{f}$
1b)	$Df \supset 2\pi i(\cdot)\hat{f}$
2a)	$\tau_a f \supset e^{-2\pi i a(\cdot)}\hat{f}$
	$e^{2\pi i a(\cdot)} f \supset \tau_a \hat{f}$

The Class S, and S'

Distributions (generalized functions) were introduced by Laurent Schwartz ($\sim$ 1940) and S Sobolev ($\sim$ 1935), to give a mathematically rigorous theory of mathematical objects like the Dirac δ-function

[note Schwarz in Cauchy-Schwarz was the German H A Schwarz]

Definition The function class S are complex valued functions f of a real variable, such that $f: \mathbb{R} \to \mathbb{C}$ satisfies

$$\sup_{x \in \mathbb{R}} \left| |x|^\alpha D^\beta f(x) \right| < \infty$$

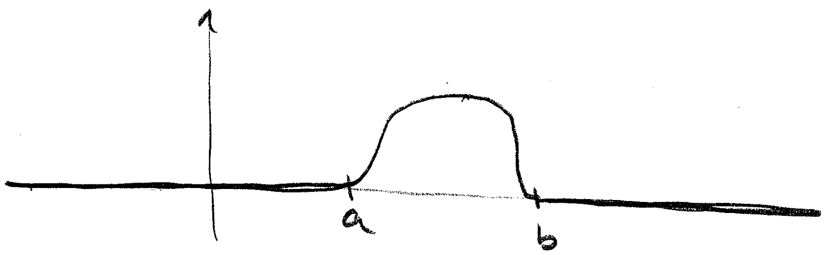
for any choice of $\alpha \geq 0$ and $\beta \geq 0$.

Ex $f(x) = e^{-ax^2}$ $(a > 0)$

(what if $a = -i$? $a < 0$?)

Ex

$$f(x) = \begin{cases} 0 & (x \leq a) \\ e^{-\frac{1}{(x-a)^2} - \frac{1}{(x-b)^2}} & (a < x < b) \\ 0 & (x \geq b) \end{cases}$$



this function f has "compact support" and is C^∞, but <u>not</u> (real) analytic (i.e., its power series is not convergent everywhere).

Ex Let $g \in L^1(\mathbb{R})$ Then

$$\int_{\mathbb{R}} g(y) \, \frac{1}{\varepsilon} f\left(\frac{x-y}{\varepsilon}\right) dy \longrightarrow g(x) \text{ when } \varepsilon \to 0$$

(in other words,

$$g * \frac{1}{\varepsilon} f\left(\frac{\cdot}{\varepsilon}\right) \longrightarrow g(x) \text{ when } \varepsilon \to 0$$

almost everywhere)

Note that the Fourier transform is well defined for $f \in S$.

<u>Properties of S</u>

<u>Lemma</u> I) if $f \in S$ and if

$$g(x) = x^\alpha D^\beta f(x) \qquad (\alpha, \beta \in \mathbb{Z}^+)$$

then $g \in S$

II) $f \in S \Rightarrow \hat{f} \in S$.

<u>Proof</u> I) just calculate

II): Let $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x \xi}\, dx$

Then $D^\beta \hat{f}(\xi) = \int_{\mathbb{R}} f(x) \left(\frac{d}{d\xi}\right)^\beta e^{-2\pi i x \xi}\, dx$

↑ this is ok! why?

$$= \int_{\mathbb{R}} \underbrace{f(x)\, (-2\pi i x)^\beta}_{\text{decays rapidly}} e^{-2\pi i x \xi}\, dx$$

$$\Rightarrow \quad |D^\beta \hat{f}(\xi)| \leq \int |f(x)|\, |2\pi x|^\beta\, dx < \infty \qquad (*)$$

Also, $2\pi i \xi \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, \underbrace{(2\pi i \xi)\, e^{-2\pi i x \xi}}_{= -\frac{d}{dx} e^{-2\pi i x \xi}}\, dx$

$$\underset{\substack{\uparrow \\ \text{partial integrat.}}}{=} \int_{-\infty}^{\infty} \frac{d}{dx} f(x)\, e^{-2\pi i x \xi}\, dx = \mathcal{F}(Df).$$

$$\Rightarrow \quad |\xi \hat{f}(\xi)| \leq \frac{1}{2\pi} \int |Df(x)|\, dx < \infty$$

this can be repeated for any power ξ^α, which together with $(*)$ concludes the result.

Ex $f(x) = e^{-\pi x^2} \Longrightarrow \hat{f} = f$

[note if $f = \mathcal{F} f$, then f is a "fix point" for $\mathcal{F}$. Are there any other fix points?]

Proof $f \in S$ and

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx$$

$$\Rightarrow \quad D\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2} (-2\pi i x) e^{-2\pi i x \xi} \, dx$$

$$= i \int_{\mathbb{R}} D\left(e^{-\pi x^2}\right) e^{-2\pi i x \xi} \, dx =$$

$$= i \, \mathcal{F}(Df) = -2\pi \xi \hat{f}(\xi)$$

Also $\hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2} dx = 1$

(check in calculus book if you don't remember)

Hence

$$\begin{cases} \hat{f}'(\xi) + 2\pi \xi \hat{f}(\xi) = 0 \\ \hat{f}(0) = 1 \end{cases} \quad \Rightarrow \quad \hat{f}(\xi) = e^{-\pi \xi^2}$$

The Fourier inversion formula

Suppose that $f \in S$, and

$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) \, dx$. Then

$$f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} \hat{f}(\xi) \, d\xi .$$

<u>Proof</u> Suppose we can check that

$$f(0) = \int_{\mathbb{R}} \hat{f}(\xi)\, d\xi \quad \text{for all } f \in S;$$

then we are done, because then

$$f(a) = \tau_a f(0) = \int_{\mathbb{R}} \mathcal{F}(\tau_a f)\, d\xi =$$

$$= \int_{\mathbb{R}} e^{2\pi i a \xi}\, \hat{f}(\xi)\, d\xi .$$

I) Assume that $f(0)=0$. Then $g(x) = \frac{f(x)}{x} \in S$,

and $f(x) = x\, g(x)$,

If $g(x) \supset \hat{g}(\xi)$ then $-2\pi i x\, g(x) \supset D\hat{g}$

That means that $\hat{f}(\xi) = -\frac{1}{2\pi i}\, \hat{g}'(\xi)$

$$\Rightarrow \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{g}'(\xi)\, d\xi = 0$$ because $\hat{g}(\xi) \in S$.

II) if $f(0) \neq 0$, consider

$$\tilde{f}(x) \equiv f(x) - f(0)\, e^{-\pi x^2} \supset \hat{f}(\xi) - f(0)\, e^{-\pi \xi^2} = \hat{\tilde{f}}(\xi)$$

Then $\tilde{f}(0) = 0 = \int_{-\infty}^{\infty} \hat{\tilde{f}}(\xi)\, d\xi =$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi - \int_{-\infty}^{\infty} f(0)\, e^{-\pi \xi^2}\, d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi - f(0) .$$

<u>done</u>