Note that if $\{\varphi_k\}_{k\in\mathbb{Z}}$ is orthonormal, then

$$\|f\|^2 = \langle \sum c_k\varphi_k, \sum \bar{c}_k \bar{\varphi}_k \rangle =$$
$$= \sum_{j\neq k} \langle \sum c_j\varphi_j, \sum c_k\varphi_k \rangle + \sum c_k\bar{c}_k \langle \varphi_k, \varphi_k \rangle = \sum |c_k|^2 .$$

But for a general basis that is not true.

<u>Def</u> A Riesz basis for a closed subspace $V \in L^2$ is a basis that satisfies

$$A\|f\|^2 \leq \sum_k |c_k|^2 \leq B\|f\|^2$$

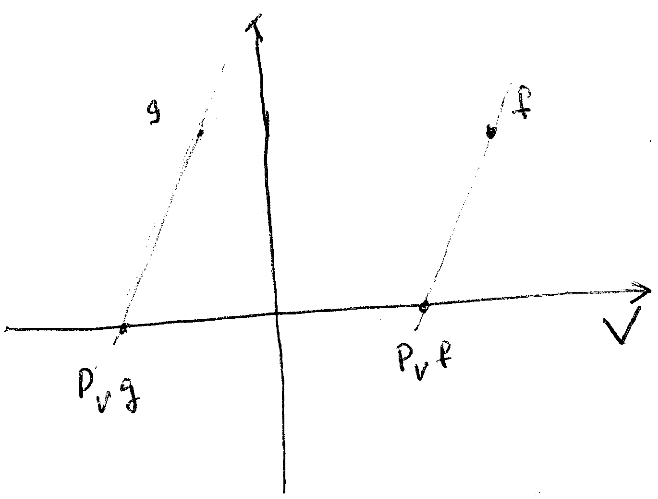
for some constants $A \leq 1 \leq B$.

<u>Note</u> If $\tilde{f}$ is an approximation of f,

$$A\|f-\tilde{f}\| \leq \sum_k |c_k-\tilde{c}_k|^2 \leq B\|f-\tilde{f}\|^2$$

A projection onto V: (orthogonal)

$$P_V f = \sum \langle f, \varphi_k \rangle \varphi_k$$



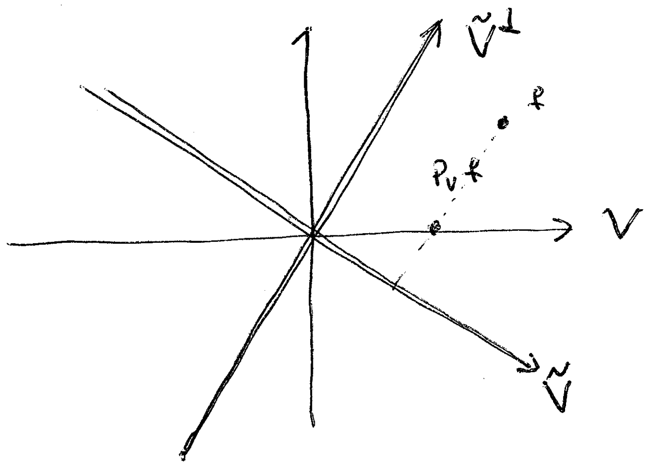
An image of a non orthogonal projection.
Here
$V = \{(x,0) \in \mathbb{R}^2\}$.

Biorthogonal bases

A biorthogonal basis is a basis $\{\varphi_k\}$ of a subspace V together with a "dual family" $\{\tilde{\varphi}_k\}$ such that

$$\langle \varphi_k, \tilde{\varphi}_n \rangle = \delta_{k,n} = \begin{cases} 1 & \text{if } k=n \\ 0 & \text{otherwise} \end{cases}$$

We let $P_V f = \sum_k \langle f, \tilde{\varphi}_k \rangle \varphi_k$

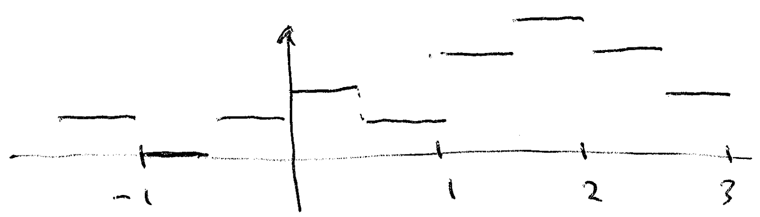


The Haar scaling function

$$\varphi(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

This can be used for piecewise constant approximation:

$$f(t) \longleftrightarrow f_1(t) = \sum_k s_{1k}\, \varphi(2t-k)$$



Definition

A <u>multi resolution analysis</u> is a family of closed subspaces $V_j \subset L^2(\mathbb{R})$ such that

1) $V_j \subset V_{j+1}$ $\quad j \in \mathbb{Z}$

2) $f \in V_j \iff f(2\cdot) \in V_{j+1}$ $\quad j \in \mathbb{Z}$

3) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$

4) $\bigcap_j V_j = \{0\}$

5) There is a scaling function $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}$ is a Riesz basis for V_0.

<u>Def</u> $\quad \varphi_{jk} = 2^{j/2} \varphi(2^j t - k)$

Here 3) means that for any $f \in L^2(\mathbb{R})$, there is a sequence of functions $f_k \in V_k$ such that $\|f_k - f\| \to 0$ when $k \to \infty$.

4) means that if $f \in V_k$, $k \in \mathbb{Z}$, then $f(t) = 0$ for all t.

Properties of the scaling function

1) $\int_{-\infty}^{\infty} \varphi(t)\,dt = 1$

2) If $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a basis for V_0, then we must have that

$$\{ 2^{1/2}\varphi(2\cdot - k)\}$$

is a basis for V_1. And because $V_0 \subset V_1$, $\varphi \in V_1$, and therefore

$$\varphi(t) = 2\sum_k h_k \varphi(2t-k) \quad , \text{ for some } \{h_k\}.$$

This is called the scaling equation.

3) Let $H(\omega) = \sum_k h_k e^{-i\omega k}$, and let $\hat{\varphi}(\omega)$ be the Fourier transform of φ.

Then

$$\hat{\varphi}(\omega) = \sum_k h_k \,\mathcal{F}\big(2\varphi(2\cdot - k)\big)(\omega)$$

$$= \sum_{k=-\infty}^{\infty} h_k e^{-ik\omega/2} \hat{\varphi}\left(\tfrac{\omega}{2}\right) = H\left(\tfrac{\omega}{2}\right)\hat{\varphi}\left(\tfrac{\omega}{2}\right) \quad *$$

From $\hat{\varphi}(0)=1$, we conclude that

$$\sum_{k=-\infty}^{\infty} h_k = 1 .$$

* and by induction

$$\hat{\varphi}(\omega) = \prod_{j\geq 0} H(\omega/2^j)$$

Example $\varphi(t) = \operatorname{sinc} t = \dfrac{\sin \pi t}{\pi t}$

$$\Rightarrow \hat{\varphi}(\omega) = \mathbb{1}_{[-\pi,\pi]}$$

Wavelets and detail spaces

In a multiresolution $\{V_j\}$,

let f be approximated by f_0 and f_1 in V_0 and V_1 respectively.

Hence $f_0 \in V_0$, $f_1 \in V_1$. But also $f_0 \in V_1$

$\Rightarrow$ $f_1 - f_0 \in V_1$.

For the Haar wavelet we had $\psi(t) = \begin{cases} 1 & 0<t<1/2 \\ -1 & 1/2<t<1/2 \\ 0 & \end{cases}$

and $\psi(t) = \frac{1}{2}\varphi_{1,0} - \frac{1}{2}\varphi_{1,1}$

<u>Def</u> For an MRA, ψ is called a <u>wavelet</u> if $W_0 \subset V_1$ is spanned by $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ and $V_1 = W_0 \oplus V_0$,

i.e., each $f_1 \in V_1$ can be written (uniquely) as

$$f_1 = f_0 + d_0 \quad \text{with } f_0 \in V_0,\ d_0 \in W_0.$$

W_0 is called a <u>detail space</u>

Properties of the wavelets

1) $\int \psi(t)\,dt = 0 \qquad (\Leftrightarrow \hat{\psi}(0) = 0)$

2) $\psi(t) \in V_1 \Leftrightarrow \psi(t) = 2\sum_k g_k\, \varphi(2t-k)$

hence $\qquad \hat{\psi}(\omega) = G(\frac{\omega}{2})\, \hat{\varphi}(\frac{\omega}{2})$,

where $\qquad G(\omega) = \sum_k g_k e^{-ik\omega}$

and because $\hat{\varphi}(0) = 1$,

$$G(0) = \sum_k g_k = 0.$$

MRA and wavelet decomposition

We have $V_1 = V_0 \oplus W_0$, where

V_0 is spanned by $\{\varphi(\cdot - k)\}$, and

W_0 is spanned by $\{\psi(\cdot - k)\}$, both of which are supposed to be Riesz bases.

Let $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ and define

W_j = linear span of $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$, i.e.

$$W_j = \{ d_j(t) = \sum_k w_{jk}\psi_{jk}(t), \quad w_{jk} \in \mathbb{C} \}$$

Let $J \geq 0$ be an integer, corresponding to the highest resolution, or in other words, the finest detail, of interest.

Take f_J be an approximation of $f \in L^2(\mathbb{R})$.

Then
$$\begin{aligned} f_J \in V_J &= W_{J-1} \oplus V_{J-1} \\ &= W_{J-1} \oplus W_{J-2} \oplus V_{J-2} \\ &= \ldots \\ &= W_{J-1} \oplus W_{J-2} \oplus W_{J-3} \oplus \ldots \oplus W_{j_0} \oplus V_{j_0}. \end{aligned}$$

Then we can write

$$\begin{aligned} f_J(t) &= d_{J-1}(t) + d_{J-2}(t) + \ldots + d_{j_0}(t) + f_{j_0}(t) \\ &= \sum_{j=j_0}^{J-1} \sum_k w_{j,k}\psi_{jk}(t) + \sum_k s_{j_0 k}\varphi_{j_0 k}(t). \end{aligned}$$

The last sum, $\sum_k s_{j_0 k}\, \varphi_{j_0 k}(t)$,

converges to zero when $j_0 \to -\infty$ because

$\bigcap V_j = \{0\}$.

Also, because $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$

one can choose $f_J \to f$ in L^2, $f_J \in V_J$.

So letting $J \to \infty$ we find

$$f(t) = \sum_{j,k} w_{j,k}\, \psi_{j,k}(t).$$

This is the <u>wavelet decomposition</u> of f.

<u>Example</u> $\varphi(t) = \operatorname{sinc} t = \dfrac{\sin \pi t}{\pi t}$

corresponding to

$$\hat{\varphi}(\omega) = \mathbb{1}_{-\pi < \omega \leq \pi}$$

Then V_0 is the set of bandlimited functions, with cutoff π, and V_j the set of bandlimited functions with cutoff $2^j \pi$.

Here $\hat{\psi}(\omega) = \mathbb{1}_{\pi < |\omega| < 2\pi}$