Note that if $\{\varphi_k\}_{k\in\mathbb{Z}}$ is orthonormal, then

$$\|f\|^2 = \langle \sum c_k \varphi_k, \sum \bar{c}_k \bar{\varphi}_k \rangle =$$
$$= \sum_{j\neq k} \langle \sum c_j \varphi_j, \sum c_k \varphi_k \rangle + \sum c_k \bar{c}_k \langle \varphi_k, \varphi_k \rangle = \sum |c_k|^2 .$$

But for a general basis that is not true.

Def A Riesz basis for a closed subspace $V \in L^2$ is a basis that satisfies

$$A\|f\|^2 \leq \sum_k |c_k|^2 \leq B\|f\|^2$$

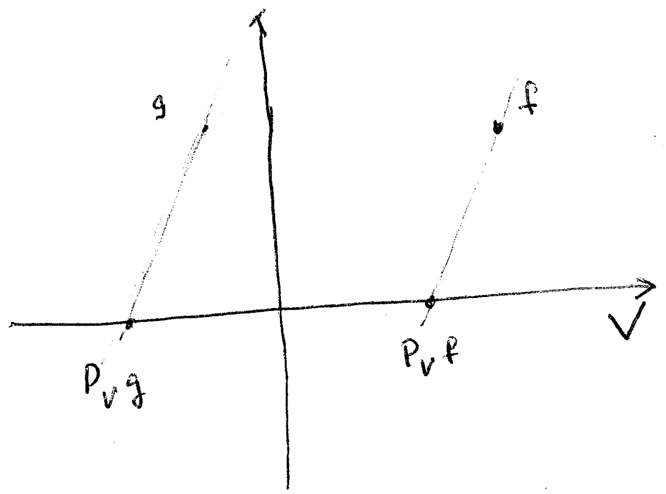
for some constants $A \leq 1 \leq B$.

Note If $\tilde{f}$ is an approximation of f,

$$A\|f-\tilde{f}\| \leq \sum_k |c_k - \tilde{c}_k|^2 \leq B\|f-\tilde{f}\|^2$$

A projection onto V: (orthogonal)

$$P_V f = \sum \langle f, \varphi_k \rangle \varphi_k$$



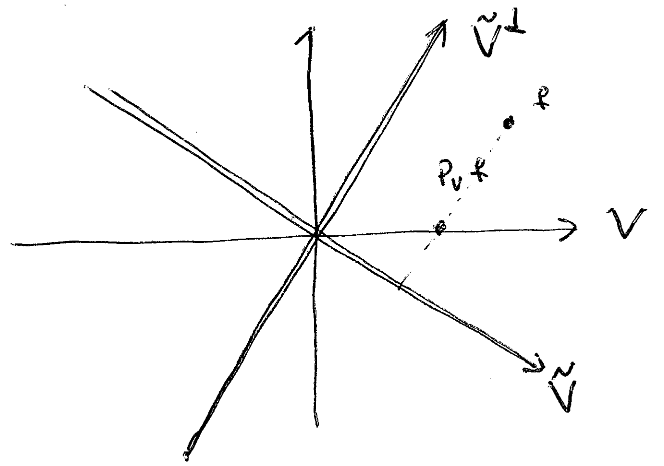
An image of a non orthogonal projection.
Here
$V = \{(x,0) \in \mathbb{R}^2\}$.

Biorthogonal bases

A biorthogal basis is a basis $\{\varphi_k\}$ of a subspace V together with a "dual family" $\{\tilde{\varphi}_k\}$ such that

$$\langle \varphi_k, \tilde{\varphi}_n \rangle = \delta_{k,n} = \begin{cases} 1 & \text{if } k=n \\ 0 & \text{otherwise} \end{cases}$$

We let $P_V f = \sum_k \langle f, \tilde{\varphi}_k \rangle \varphi_k$

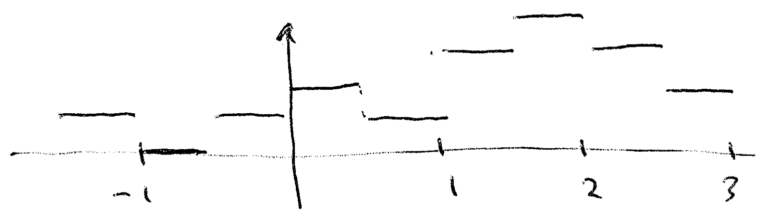


The Haar scaling function

$$\varphi(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

This can be used for piecewise constant approximation:

$$f(t) \longmapsto f_1(t) = \sum_k s_{1k} \varphi(2t-k)$$



Definition

A <u>multi resolution analysis</u> is a family of closed subspaces $V_j \subset L^2(\mathbb{R})$ such that

1) $V_j \subset V_{j+1} \qquad j \in \mathbb{Z}$

2) $f \in V_j \iff f(2\cdot) \in V_{j+1} \qquad j \in \mathbb{Z}$

3) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$

4) $\bigcap_j V_j = \{0\}$

5) There is a scaling function $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}$ is a Riesz basis for V_0.

<u>Def</u> $\quad \varphi_{jk} = 2^{j/2}\varphi(2^j t - k)$

Here 3) means that for any $f \in L^2(\mathbb{R})$, there is a sequence of functions $f_k \in V_k$ such that $\| f_k - f \| \to 0$ when $k \to \infty$.

4) means that if $f \in V_k$, $k \in \mathbb{Z}$, then $f(t) = 0$ for all t.

Properties of the scaling function

1) $\int_{-\infty}^{\infty} \varphi(t)\,dt = 1$

2) If $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a basis for V_0, then we must have that

$$\{ 2^{1/2}\varphi(2\cdot - k)\}$$

is a basis for V_1. And because $V_0 \subset V_1$, $\varphi \in V_1$, and therefore

$$\varphi(t) = 2\sum_k h_k\, \varphi(2t-k) \quad , \text{ for some } \{h_k\}.$$

This is called the scaling equation.

3) Let $H(\omega) = \sum_k h_k e^{-i\omega k}$, and let $\hat{\varphi}(\omega)$ be the Fourier transform of φ.

Then

$$\hat{\varphi}(\omega) = \sum_k h_k\, \mathcal{F}\big(2\varphi(2\cdot - k)\big)(\omega)$$

$$= \sum_{k=-\infty}^{\infty} h_k e^{-ik\omega/2}\, \hat{\varphi}\left(\tfrac{\omega}{2}\right) = H\left(\tfrac{\omega}{2}\right)\hat{\varphi}\left(\tfrac{\omega}{2}\right) \quad *$$

From $\hat{\varphi}(0) = 1$, we conclude that

$$\sum_{k=-\infty}^{\infty} h_k = 1 .$$

* and by induction

$$\hat{\varphi}(\omega) = \prod_{j\geq 0} H(\omega/2^j)$$

Example $\varphi(t) = \operatorname{sinc} t = \frac{\sin \pi t}{\pi t}$

$$\Rightarrow \hat{\varphi}(\omega) = \mathbb{1}_{[-\pi,\pi]}$$

Wavelets and detail spaces

In a multiresolution $\{V_j\}$,

let f be approximated by f_0 and f_1 in V_0 and V_1 respectively.

Hence $f_0 \in V_0$, $f_1 \in V_1$. But also $f_0 \in V_1$

$\Rightarrow$ $f_1 - f_0 \in V_1$.

For the Haar wavelet we had $\psi(t) = \begin{cases} 1 & 0<t<1/2 \\ -1 & 1/2<t<1 \\ 0 & \end{cases}$

and $\psi(t) = \frac{1}{2}\varphi_{1,0} - \frac{1}{2}\varphi_{1,1}$

<u>Def</u> For an MRA, ψ is called a <u>wavelet</u> if $W_0 \subset V_1$ is spanned by $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ and $V_1 = W_0 \oplus V_0$,

i.e., each $f_1 \in V_1$ can be written (uniquely) as

$$f_1 = f_0 + d_0 \quad \text{with } f_0 \in V_0,\ d_0 \in W_0.$$

W_0 is called a <u>detail space</u>

Properties of the wavelets

1) $\int \psi(t)\,dt = 0 \qquad (\Leftrightarrow \hat{\psi}(0) = 0)$

2) $\psi(t) \in V_1 \Leftrightarrow \psi(t) = 2\sum_k g_k\,\varphi(2t-k)$

hence $\hat{\psi}(\omega) = G(\frac{\omega}{2})\,\hat{\varphi}(\frac{\omega}{2})$,

where $G(\omega) = \sum_k g_k e^{-ik\omega}$

and because $\hat{\varphi}(0) = 1$,

$$G(0) = \sum_k g_k = 0.$$

MRA and wavelet decomposition

We have $V_1 = V_0 \oplus W_0$, where

V_0 is spanned by $\{\varphi(\cdot - k)\}$, and

W_0 is spanned by $\{\psi(\cdot - k)\}$, both of which are supposed to be Riesz bases.

Let $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ and define

W_j = linear span of $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$, i.e.

$$W_j = \{ d_j(t) = \sum_k w_{jk}\psi_{jk}(t), \quad w_{jk} \in \mathbb{C} \}$$

Let $J \geq 0$ be an integer, corresponding to the highest resolution, or in other words, the finest detail, of interest.

Take f_J be an approximation of $f \in L^2(\mathbb{R})$.

Then
$$\begin{aligned} f_J \in V_J &= W_{J-1} \oplus V_{J-1} \\ &= W_{J-1} \oplus W_{J-2} \oplus V_{J-2} \\ &= \dots \\ &= W_{J-1} \oplus W_{J-2} \oplus W_{J-3} \oplus \dots \oplus W_{j_0} \oplus V_{j_0}. \end{aligned}$$

Then we can write

$$\begin{aligned} f_J(t) &= d_{J-1}(t) + d_{J-2}(t) + \dots + d_{j_0}(t) + f_{j_0}(t) \\ &= \sum_{j=j_0}^{J-1}\sum_k w_{j,k}\psi_{j,k}(t) + \sum_k s_{j_0 k}\varphi_{j_0 k}(t). \end{aligned}$$

The last sum, $\sum_k s_{j_0 k}\, \varphi_{j_0 k}(t)$,

converges to zero when $j_0 \to -\infty$ because

$\bigcap V_j = \{0\}$.

Also, because $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$

one can choose $f_J \to f$ in L^2, $f_J \in V_J$.

So letting $J \to \infty$ we find

$$f(t) = \sum_{j,k} w_{j,k}\, \psi_{j,k}(t).$$

This is the <u>wavelet decomposition</u> of f.

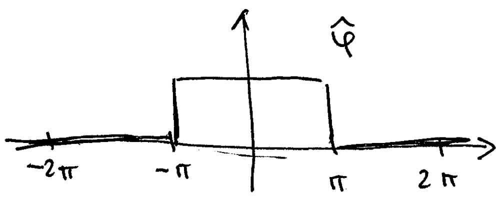
<u>Example</u> $\quad \varphi(t) = \operatorname{sinc} t = \dfrac{\sin \pi t}{\pi t}$

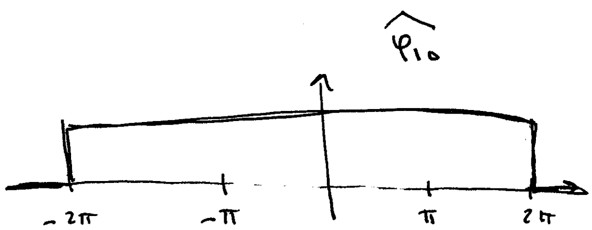
corresponding to

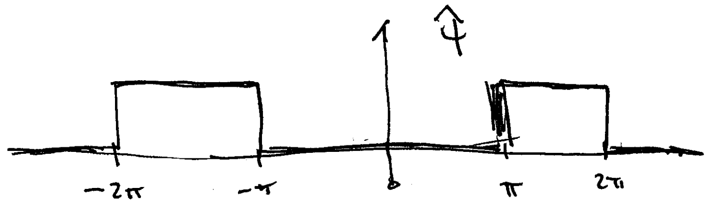
$$\hat{\varphi}(\omega) = \mathbb{1}_{-\pi < \omega \le \pi}$$

Then V_0 is the set of bandlimited functions, with cutoff π, and V_j the set of bandlimited functions with cutoff $2^j \pi$.

Here $\hat{\psi}(\omega) = \mathbb{1}_{\pi < |\omega| < 2\pi}$







Orthogonal Wavelet Decomposition

General properties of scaling function φ and wavelet ψ:

$$\varphi(t) = \sum_k h_k \, 2\varphi(2t-k) \qquad \int \varphi \, dt = 1$$

$$\hat{\varphi}(\omega) = H\left(\tfrac{\omega}{2}\right) \varphi\left(\tfrac{\omega}{2}\right) \quad \text{where} \quad H(\omega) = \sum h_k e^{-i\omega k} \qquad \hat{\varphi}(0)=0$$

$$H(0) = 1 \iff \sum h_k = 1$$

$$\psi(t) = \sum_k g_k \, 2\varphi(2t-k) \qquad \int \psi(t)\,dt = 0$$

$$\hat{\psi}(\omega) = G\left(\tfrac{\omega}{2}\right) \hat{\varphi}\left(\tfrac{\omega}{2}\right) \; ; \quad G(0)=0, \quad G(\omega) = \sum g_k e^{-ik\omega}.$$

NB The decomposition $V_{j+1} = V_j \oplus W_j$. It is defined by the filter G. However, there is only one way of writing $V_{j+1} = V_j \oplus W_j$ so that

$$\langle v, w \rangle = 0 \quad \text{for all} \quad v \in V_j \quad w \in W_j$$

For an orthonormal system we require

$$\int_{-\infty}^{\infty} \varphi(t-k) \overline{\varphi(t-n)} \, dt = 0 \quad \text{when } n \neq k$$
$$= 1 \quad \text{when } n = k.$$

In the scaling equation:

$$\int \varphi(t-n) \overline{\varphi(t-m)} \, dt =$$

$$= 4 \sum_{k,k'} h_k h_{k'} \int \varphi(2(t-n)-k) \overline{\varphi(2(t-m)-k')} \, dt$$

$$= 2 \sum_{k,k'} h_k h_{k'} \underbrace{\int \varphi(t-2n-k) \overline{\varphi(t-2m-k')} \, dt}_{= \begin{cases} 1 & \text{if } 2n+k = 2m+k' \\ 0 & \text{otherwise} \end{cases}}$$

$$= 2 \sum_{2n+k=2m+k'} h_k h_{k'}$$

without loss of generality, we may take $n=0$

$$\Rightarrow \quad \int \varphi(t)\varphi(t-m)\,dt = 2\sum_{k=2m+k'} h_k h_{k'} = 2\sum_{k'} h_{k'+2m} h_{k'}.$$

So $$2\sum_k h_k h_{k+2m} = \begin{cases} 1 & \text{if } m=0 \\ 0 & m\neq 0 \end{cases}.$$ (a)

Next we demand that $\{\psi(t-k)\}_{k=-\infty}^{\infty}$ is an ON-basis for W_0.

$$\int_{-\infty}^{\infty} \psi(t-k)\overline{\psi(t-m)}\,dt = \begin{cases} 1 & \text{if } k=m \\ 0 & \text{otherwise} \end{cases}$$

$$\Updownarrow$$

$$2\sum_k g_k g_{2m+k} = \begin{cases} 1 & \text{if } m=0 \\ 0 & \text{otherwise} \end{cases}$$ (b)

And finally we demand $V_0 \perp W_0$, which means that

$$\int_{-\infty}^{\infty} \varphi(t-k)\overline{\psi(t-n)}\,dt = 0 \quad \text{for all } k, n,$$

which means that

$$\sum_m h_{m+2k}\, g_{m+2n} = 0$$ (c)

The Haar wavelet is an example showing that such systems exist.

Once constructed, such a system can be used to approximate $f \in L^2(\mathbb{R})$:

$$f \approx f_j + d_j \qquad \text{where} \quad f_j = P_j f = \sum_k \langle f, \varphi_{jk}\rangle \varphi_{jk}$$

$$d_j = Q_j f = \sum_k \langle f, \psi_{jk}\rangle \psi_{jk}.$$

Proposition

The ON-condition implies that

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1$$

$$|G(\omega)|^2 + |G(\omega+\pi)|^2 = 1$$

$$H(\omega)\overline{G(\omega)} + H(\omega+\pi)\overline{G(\omega+\pi)} = 0.$$

Proof of the first:

$$H(\omega)\overline{H(\omega)} + H(\omega+\pi)\overline{H(\omega+\pi)} = \quad \text{(assuming } h_k \text{ real)}$$

$$= \Big(\sum_k h_k e^{-i\omega k}\Big)\Big(\sum_{k'} h_{k'} e^{i\omega k'}\Big) + \Big(\sum_k h_k(-1)^k e^{-i\omega k}\Big)\Big(\sum_{k'} h_{k'}(-1)^{k'} e^{i\omega k'}\Big)$$

$$= \sum_{k\,k'} h_k h_{k'} \left(1 + (-1)^{k+k'}\right) e^{-i\omega(k-k')}$$

$$= 2 \sum_{k+k'=2m} h_k h_{k'} e^{-i\omega(k-k')} = \left\{\begin{array}{l} u = k-k' \\ k' = k-u \end{array}\right.$$

$$= 2 \sum_{2k-u=2m} h_k h_{k-u} e^{-i\omega u} = \{ \text{need } u = 2u' \}$$

$$= 2 \sum_{k,u'} h_k h_{k-2u'} e^{-i\omega 2u'} = \sum_{u'} e^{-i\omega 2u'} \underbrace{2\sum_k h_k h_{k-2u'}}_{= \left\{\begin{array}{ll} 1 & \text{if } u'=0 \\ 0 & \text{otherwise} \end{array}\right.} = 1.$$

Prop (conditions on the scaling function and the wavelet)

1) $\sum |\hat{\varphi}(\omega + 2\pi k)|^2 = 1$

2) $\lim_{j\to -\infty} \hat{\varphi}(2^{-j}\omega) = 1$ (obvious if $\hat{\varphi}$ continuous at $\omega=0$)

3) $\hat{\varphi}(2\omega) = H(\omega)\hat{\varphi}(\omega)$; H 2π-periodic

4) $\sum_j |\hat{\psi}(2^j\omega)|^2 = 1$

5) $\sum_{j\geq 0} \hat{\psi}(2^j\omega)\overline{\hat{\psi}(2^j(\omega+2k\pi))} = 0$ when k is odd

6) $\sum_{j\geq 0}\sum_k |\hat{\psi}(2^j(\omega+2\pi k))|^2 = 1.$

Proof of 1)

$$\int_{\mathbb{R}} \varphi(t)\,\overline{\varphi(t-k)}\,dt = \begin{cases} 0 & k\neq 0 \\ 1 & k=0 \end{cases} = \delta_k$$

$$\Leftrightarrow \quad \delta_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\xi)\,\overline{e^{-i\omega k}\,\hat{\varphi}(\xi)}\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\xi)\,e^{i\omega k}\,\overline{\hat{\varphi}(\xi)}\,d\xi =$$

$$= \sum_n \frac{1}{2\pi}\int_0^{2\pi} |\hat{\varphi}(\xi+2\pi n)|^2\, e^{i(\xi+2\pi n)k}\,d\xi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i\xi k}\sum_n |\hat{\varphi}(\xi+2\pi n)|^2\,d\xi$$

So δ_k is the Fourier series of the periodic function $r(\omega)=\sum_{n=-\infty}^{\infty} |\hat{\varphi}(\omega+2\pi n)|^2$, and that implies that $r(\omega)=1$.

The continuous wavelet transform

We have seen how to write $f\in L^2(\mathbb{R})$ as a "wavelet series",

$$f = \sum_{jk} c_{jk}\,\psi_{jk}.$$

There is a different kind of decomposition.

Take $\psi\in L^2(\mathbb{R})$, and assume that

$$C_\psi \equiv \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{|\xi|}|\hat{\psi}(\xi)|^2\,d\xi < \infty.$$

If $\psi\in L^1(\mathbb{R})$, then $\hat{\psi}(\xi)$ is continuous, and then we must have $\hat{\psi}(0)=\int_{\mathbb{R}}\psi(x)\,dx = 0$.

$\ulcorner$ the continuity of $\hat{\psi}$ follows from

$$|\hat{\psi}(\xi+h)-\hat{\psi}(\xi)| = \left| \int_{-\infty}^{\infty} \left(e^{-i(\xi+h)x} - e^{-i\xi x}\right) f(x)\,dx \right|$$

$$\leq \int_{-\infty}^{\infty} |e^{-ihx}-1|\,|f(x)|\,dx = \int_{|x|<M} |e^{-ihx}-1|\,|f(x)|\,dx + \int_{|x|>M} |f(x)|\,dx .$$

Given $\varepsilon>0$, take M so large that $\int_{|x|>M} |f(x)|\,dx < \frac{\varepsilon}{2}$, and

then δ so small that

$$|1-e^{-ihx}| < \varepsilon \left(2 \int |f(x)|\,dx\right)^{-1}$$

when $|x|<M$, $|h|<\delta$

We then find that $|\hat{\psi}(\xi+h)-\hat{\psi}(\xi)| < \varepsilon$ when $|h|<\delta$. $\lrcorner$

Define $\psi^{ab}(x) = |a|^{-1/2}\,\psi\left(\frac{x-b}{a}\right)$

(this is a doubly indexed family of wavelets)

Definition

$$(T^{wav} f)(a,b) = \langle f, \psi^{ab} \rangle = \int f(x)\, \overline{\psi\left(\frac{x-b}{a}\right)}\, |a|^{-1/2}\, dx$$

Note Because $\|\psi^{ab}\|_{L^2} = 1$, $|(T^{wav} f)(a,b)| \leq \|f\|_{L^2}$.

Proposition (The resolution of the identity)

For all $f, g \in L^2(\mathbb{R})$, the following equality holds:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (T^{wav} f)(a,b)\, \overline{(T^{wav} g)(a,b)}\, \frac{1}{|a|^2}\, da\, db = C_\psi \langle f, g\rangle$$

(this result should be compared with

$$\int f(x)\,\overline{g(x)}\,dx = \frac{1}{2\pi} \int \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,d\xi \qquad)$$.

we can rewrite the equality as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (T^{wav} f)(a,b) \overline{\int_{-\infty}^{\infty} g(x)\, |a|^{-1/2}\, \psi\left(\frac{x-b}{a}\right) dx} \frac{1}{|a|^2}\, da\, db$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (T^{wav} f)(a,b)\, \psi\left(\frac{x-b}{a}\right) \frac{1}{|a|^{2+\frac{1}{2}}}\, da\, db \right) \overline{g(x)}\, dx$$

$$= \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (T^{wav} f)(a,b)\, \psi^{ab}(x) \frac{1}{|a|^2}\, da\, db \right)} \overline{g(x)}\, dx \quad = \langle f, g \rangle$$

so in "a weak sense" this must be equal to f.

Proof

$$\iint \frac{da\, db}{|a|^2} (T^{wav} f)(a,b)\, \overline{(T^{wav} g)(a,b)}$$

$$= \iint \frac{da\, db}{|a|^2} \left[\frac{1}{2\pi} \int d\xi\, \hat{f}(\xi)\, |a|^{1/2} e^{-ib\xi}\, \overline{\hat{\psi}(a\xi)} \right] \times$$
$$\times \left[\frac{1}{2\pi} \int d\xi'\, \overline{\hat{g}(\xi')}\, |a|^{1/2} e^{ib\xi'}\, \hat{\psi}(a\xi') \right] \qquad (*)$$

i.e. we have expressed $(T^{wav} f)(a,b)$ using parseval's formula on

$$T^{wav} f(a,b) = \int_{-\infty}^{\infty} f(x)\, |a|^{-1/2}\, \psi\left(\frac{x-b}{a}\right) dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\, \overline{\mathcal{F}\left(|a|^{-1/2}\psi\left(\frac{\cdot - b}{a}\right)\right)(\xi)}\, d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\, |a|^{1/2} e^{ib\xi}\, \hat{\psi}(a\xi)\, d\xi \,.$$

Next we note that

$$\int_{-\infty}^{\infty} \underbrace{\hat{f}(\xi)\, |a|^{1/2}\, \overline{\hat{\psi}(a\xi)}}_{\equiv F_a(\xi)}\, e^{-ib\xi}\, d\xi = \widehat{F_a}(b),$$

where ξ and b are treated as dual variables in a Fourier transform.

Similarly

$$\int e^{ib\xi'} \underbrace{|a|^{1/2}\, \overline{\hat{g}(\xi')}\, \hat{\psi}(a\xi')}_{\equiv\ \overline{G_a(\xi)}}\, d\xi' = \hat{G}_a(b).$$

Therefore we can write

$$\circledast = \frac{1}{4\pi^2} \int \frac{da}{a^2} \int db\, \hat{F}_a(b)\, \overline{\hat{G}_a(b)}$$

$$= \frac{1}{2\pi} \int \frac{da}{a^2} \int F_a(\xi)\, \overline{G_a(\xi)}\, d\xi$$

$$= \frac{1}{2\pi} \int \frac{da}{a} \int \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, |\hat{\psi}(a\xi)|^2\, d\xi$$

$$= \frac{1}{2\pi} \int \hat{f}(\xi)\, \overline{\hat{g}(\xi)} \underbrace{\int \frac{|\hat{\psi}(a\xi)|^2}{|a|}\, da}_{= \int |\hat{\psi}(a)|^2 \frac{da}{a} = c_\psi}\, d\xi$$

$$= c_\psi \frac{1}{2\pi} \int \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, d\xi = c_\psi \langle f, g \rangle.$$

(note that all integrals of the form

$$\int \frac{F(xy)}{y}\, dy$$ are independent of

x, if the integral is convergent).

■

The family $\{\psi^{ab}\}_{\substack{a\in\mathbb{R}\\ b\in\mathbb{R}}}$ is uncountable,

but in the discrete wavelet case, we have only countably many ψ_{jk}.

How many do we really need?

Definition

A family of functions $(\varphi_j)_{j\in J}$ in a Hilbert space $\mathcal{H}$ is called a frame if there exist $0 < A \leq B$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, \varphi_j\rangle|^2 \leq B\|f\|^2 .$$

A and B are called the frame bounds.

A frame is called tight if $A = B$

Proposition If $(\varphi_j)_{j\in J}$ is a tight frame with $A=1$ and $\|\varphi_j\| = 1$ for all $j \in J$, then (φ_j) is an ON-basis.

Proof If $\langle f, \varphi_j\rangle = 0$ for all $j \in J$, then $\|f\| = 0$ and therefore $(\varphi_j)_{j\in J}$ span all of $\mathcal{H}$.

(otherwise let $\mathcal{H}_0$ be the span of $(\varphi_j)_{j\in J}$ and let $\mathcal{H}_1$ be the orthogonal complement to $\mathcal{H}_0$ in $\mathcal{H}$, i.e. $\mathcal{H}_0 \perp \mathcal{H}_1$ and $\mathcal{H}_0 \oplus \mathcal{H}_1 = \mathcal{H}$.
Take $g \in \mathcal{H}_1$, $\|g\| = 1$. But then $\langle g, \varphi_j\rangle = 0$ for all $j \in J$, which contradicts the tightness of the frame).

Also $$\|\varphi_j\|^2 = \sum_{j\in J} |\langle \varphi_j, \varphi_{j'}\rangle|^2 = \|\varphi_j\|^4 + \sum_{j' \neq j} |\langle \varphi_j, \varphi_{j'}\rangle|^2$$

But $\|\varphi_j\|^2 = \|\varphi_j\|^4 = 1 \Rightarrow \sum_{j'\neq j} |\langle \varphi_j, \varphi_{j'}\rangle|^2 = 0$,

so $\langle \varphi_j, \varphi_{j'}\rangle = 0$ if $j \neq j'$.

If a frame is tight,

$$\sum_{j \in J} |\langle f, \varphi_j \rangle|^2 = A \|f\|^2$$

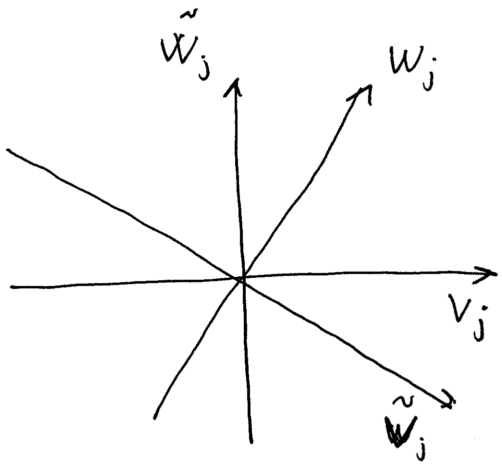
$$\Rightarrow \quad A \langle f, g \rangle = \sum_{j \in J} \langle f, \varphi_j \rangle \langle \varphi_j, g \rangle$$

$$\Rightarrow \quad f = \frac{1}{A} \sum \langle f, \varphi_j \rangle \varphi_j .$$

For a frame that is not an ON-basis, there may be many ways of writing $f = \sum c_j \varphi_j$.

The most economical way is by aid of the "dual frame" $\tilde{\varphi}_j$.

Bi orthogonal Systems



The figure indicates how V_j and W_j together span $\mathbb{R}^2$, which serves as an image of V_{j+1}.

But they are not orthogonal.

Instead there is a space $\tilde{W}_j$ that is the orthogonal complement to V_j.

$\tilde{W}_j$ is the detail space of a "dual MRA",

$\{\tilde{V}_j\}_{j=-\infty}^{\infty}$; $\tilde{V}_j \subset \tilde{V}_{j+1}$ ....

There is a dual scaling function $\tilde{\varphi}$ that satisfies a scaling equation

$$\tilde{\varphi}(t) = 2 \sum \tilde{h}_k \tilde{\varphi}(2t-k)$$

and a dual mother wavelet $\tilde{\psi}$ that satisfies

$$\tilde{\psi}(t) = 2 \sum \tilde{g}_k \tilde{\varphi}(2t-k) .$$

The conditions for <u>biorthogonality</u> are

$$\begin{cases} \langle \varphi_{j,k}, \tilde{\varphi}_{j,m} \rangle = \delta_{k,m} \\ \langle \psi_{j,k}, \tilde{\psi}_{j,m} \rangle = \delta_{k,m} \\ \langle \varphi_{j,k}, \tilde{\psi}_{j,m} \rangle = 0 \\ \langle \tilde{\varphi}_{j,k}, \psi_{j,m} \rangle = 0 \end{cases}$$

You can then derive orthogonality conditions for H, G, $\tilde{H}$ and $\tilde{G}$:

$$\begin{cases} \tilde{H}(\omega)\overline{H(\omega)} + \tilde{H}(\omega+\pi)\overline{H(\omega+\pi)} = 1 \\ \tilde{G}(\omega)\overline{G(\omega)} + \tilde{G}(\omega+\pi)\overline{G(\omega+\pi)} = 1 \\ \tilde{G}(\omega)\overline{H(\omega)} + \tilde{G}(\omega+\pi)\overline{H(\omega+\pi)} = 0 \\ \tilde{H}(\omega)\overline{G(\omega)} + \tilde{H}(\omega+\pi)\overline{G(\omega+\pi)} = 0 \end{cases}$$