Mathematics Chalmers & GU

TMA462/MMA410: Fourier and Wavelet Analysis, 2015–04–18; kl 14:00-18:00.

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Course Books: Bergh et al, Bracewell, Lecture Notes and Calculator are allowed (no solutions).
Each problem gives max 6p. Breakings: **3**: 12-17p, **4**: 18-23p och **5**: 24p-
For GU students **G** :12-20p, **VG**: 21p- (if applicable)
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma462/1415/index.html

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1. Describe the operator
$$(\uparrow 2)(\downarrow 2)(\uparrow 2)(\downarrow 2).$$

2. If $\varphi_1(x)$ and $\varphi_2(x)$ satisfy dilation equations, does their product $P(x) = \varphi_1(x)\varphi_2(x)$ satisfy a dilation equation?

3. For the scaling function $\varphi(x)$ we have the dilation equation
$$\varphi(x) = 2\sum_{k=0}^{N} h_k \varphi(2x-k),$$
written in Fourier mod as:
$$\hat{\varphi}(\omega) = H(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}).$$
Show that if $H(\omega)$ has p zeros at $\omega = \pi$, then $\hat{\varphi}(\omega)$ har p zeros at $\omega = 2\pi n$, for each $n \neq 0$.

4. The inner products $\mathbf{a}(k)$ are the Fourier coefficients of the 2π-periodic function $A(\omega)$:
$$\mathbf{a}(k) = \int_{-\infty}^{\infty} \phi(t)\overline{\phi(t-k)}\, dt \quad \text{transforms to} \quad A(\omega) = \sum_{-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi n)|^2.$$

5. The Hilbert transform of a function f is defined viz
$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy,$$
and note that for $y/x < 1$,
$$\frac{1}{x-y} = \frac{1}{x(1-\frac{y}{x})} = \frac{1}{x}\sum_{k=0}^{\infty}\Big(\frac{y}{x}\Big)^k.$$
a) How fast does the Hilbert transform Ψ of a function ψ decay as $|x| \to \infty$, when ψ is a Haar wavelet? Hint:
$$1/(x-y) \approx (1 + y/x)/x, \qquad \text{when} \quad |y/x| << 1.$$
b) How fast does Ψ decay when ψ is a wavelet with compact support and N vanishing moments.

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Lösningar/Solutions.

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1. Describe the operator

$$(\uparrow 2)(\downarrow 2)(\uparrow 2)(\downarrow 2).$$

Solution: The easiest approach would be to solve the problem using the fact that

$$(\downarrow 2)(\uparrow 2)x(n) = x(x)$$

Thus

$$\begin{aligned}\hat{x}(n) &= (\uparrow 2)(\downarrow 2)(\uparrow 2)(\downarrow 2)x(x)\\ &= (\uparrow 2)\Big((\downarrow 2)(\uparrow 2)\Big)\Big((\downarrow 2)x(x)\Big)\\ &= (\uparrow 2)(\downarrow 2)x(x)\\ &= \frac{1}{2}(1+(-1)^n)x(n).\end{aligned}$$

In the frequency domain, this corresponds to

$$\begin{aligned}\hat{X}(z) &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\Big(x(n)z^{-n} + x(n)z^{n}\Big)\\ &= \frac{1}{2}\Big(X(z) + X(-z)).\end{aligned}$$

2. If $\varphi_1(x)$ and $\varphi_2(x)$ satisfy dilation equations, does their product $P(x) = \varphi_1(x)\varphi_2(x)$ satisfy a dilation equation?

Solution: Let the two scaling functions satisfy refinement equations of the form:

$$\begin{aligned}\varphi_1(x) &= 2\sum_k h_{1,k}\varphi_1(2x-k)\\ \varphi_2(x) &= 2\sum_l h_{2,l}\varphi_2(2x-l).\end{aligned}$$

We expand the product of $\varphi_1(x)$ and $\varphi_2(x)$ viz

$$\begin{aligned}P(x) =&4\sum_k\sum_l h_{1,k}h_{2,l}\varphi_1(2x-k)\varphi_2(2x-l)\\ =&2\sum_k\Big(h_{1,k}h_{2,k}\Big)P(2x-k) + 4\sum_k\sum_{l\neq k} h_{1,k}h_{2,l}\varphi_1(2x-k)\varphi_2(2x-l),\end{aligned}$$

which implies that the product $P(x)$ does not necessarily satisfy a dilation equation in general, since the second term in the above summation need not vanish for any two scaling functions.

3. For the scaling function $\varphi(x)$ we have the dilation equation

$$\varphi(x) = 2\sum_{k=0}^{N} h_k\varphi(2x-k),$$

written in Fourier mod as:

$$\hat{\varphi}(\omega) = H(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}). \tag{1}$$

Show that if $H(\omega)$ has p zeros at $\omega = \pi$, then $\hat{\varphi}(\omega)$ har p zeros at $\omega = 2\pi n$, for each $n \neq 0$.

Solution: Iterating equation (1) we obtain the relation

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H\Big(\frac{\omega}{2^j}\Big). \tag{2}$$

By differentiating the infinite product formula(2) we end up with

$$\frac{d^i}{d\omega^i}\hat{\varphi}(\omega) = \sum_{k=1}^{\infty} \frac{1}{2^{k+i}} \frac{d^i}{d\omega^i} H\Big(\frac{\omega}{2^k}\Big) \prod_{k\neq j=1}^{\infty} H\Big(\frac{\omega}{2^j}\Big).$$

Substituting $\omega = 2^l(2m+1)\pi$, $l \geq 0$ we get

$$\frac{d^i}{d\omega^i}\hat{\varphi}(\omega)\Big|_{2^l(2m+1)\pi} = \sum_{k=1}^{\infty} \frac{1}{2^{k+i}} \frac{d^i}{d\omega^i} H(\omega)\Big|_{2^{l-k}\pi} \prod_{k\neq j=1}^{\infty} H(\omega)|_{2^{l-j}\pi}.$$

Now for $0 \leq i \leq p$, all terms in the summation on the rhs vanish since $H(\omega)$ has p zeros at $\omega = \pi$, i.e., $\hat{\varphi}(\omega)$ har p zeros at $\omega = 2\pi n$, $n > 0$.

4. The inner products $\mathbf{a}(k)$ are the Fourier coefficients of the 2π-periodic function $A(\omega)$:

$$\mathbf{a}(k) = \int_{-\infty}^{\infty} \phi(t)\overline{\phi(t-k)}\, dt \quad \text{transforms to} \quad A(\omega) = \sum_{-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi n)|^2.$$

Solution: By Parseval's identity, an inner product in the time domain equals an inner product in the frequency domain. The inner product in the time domain is between $\phi(t)$ and $\phi(t-k)$. The Fourier transforms of these functions are $\hat{\phi}(\omega)$ and $e^{-i\omega k}\hat{\phi}(\omega)$, respectively. Each inner product integrate one function times the complex conjugate of the other:

$$\begin{aligned}\mathbf{a}(k) &= \int_{-\infty}^{\infty} \phi(t)\overline{\phi(t-k)}\, dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(\omega)\overline{\hat{\phi}(\omega)} e^{i\omega k}\, d\omega \\ &= \frac{1}{2\pi}\int_{0}^{2\pi} \sum_{-\infty}^{\infty} |\hat{\phi}(\omega + 2\pi n)|^2 e^{i\omega k}\, d\omega.\end{aligned}$$

The last integral split $(-\infty, \infty)$ into an infinite nimber of 2π-pieces, using the periodicity of $e^{i\omega k}$. This integral defines the k^{th} Fourier coefficient of $A(\omega)$. Thus $A(\omega) = \sum_{-\infty}^{\infty} \mathbf{a}(k) e^{i\omega k}$.

5. The Hilbert transform of a function f is defined viz

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy,$$

and note that for $y/x < 1$,

$$\frac{1}{x-y} = \frac{1}{x(1-\frac{y}{x})} = \frac{1}{x}\sum_{k=0}^{\infty}\Big(\frac{y}{x}\Big)^k.$$

a) How fast does the Hilbert transform Ψ of a function ψ decay as $|x| \to \infty$, when ψ is a Haar wavelet? Hint:

$$1/(x-y) \approx (1 + y/x)/x, \qquad \text{when} \quad |y/x| << 1.$$

b) How fast does Ψ decay when ψ is a wavelet with compact support and N vanishing moments.

Solution: a) We have that

$$\psi(x) = \begin{cases} 1, & 0 < x < 1/2 \\ -1, & 1/2 < x < 1 \\ 0, & \text{else.} \end{cases}$$

Thus $|x| > 1$ yields

$$\Psi(x) = \frac{1}{\pi}\Big(\int_0^{1/2} \frac{1}{x-y}\,dy - \int_{1/2}^{1} \frac{1}{x-y}\,dy\Big) = \frac{1}{\pi}\frac{1}{x}\Big(\int_0^{1/2} \frac{1}{1-y/x}\,dy - \int_{1/2}^{1} \frac{1}{1-y/x}\,dy\Big)$$

$$= \frac{1}{\pi x}\Big(\int_0^{1/2} \sum_{n=0}^{\infty}\Big(\frac{y}{x}\Big)^n\,dy - \int_{1/2}^{1} \sum_{n=0}^{\infty}\Big(\frac{y}{x}\Big)^n\,dy\Big)$$

$$= \frac{1}{\pi x}\sum_{n=0}^{\infty}\frac{1}{x^n}\Big(\int_0^{1/2} y^n\,dy - \int_{1/2}^{1} y^n\,dy\Big) \approx \frac{1}{\pi x^2}\Big(\int_0^{1/2} y\,dy - \int_{1/2}^{1} y\,dy\Big)$$

$$= \frac{1}{\pi x^2}\Big[\frac{1}{2}\Big(\frac{1}{2}\Big)^2 - \frac{1}{2}\Big(1\Big(\frac{1}{2}\Big)^2\Big)\Big] = \frac{1}{\pi x^2}\Big(-\frac{1}{4}\Big) = -\frac{1}{4\pi x^2}.$$

b)

$$\Psi(x) = \frac{1}{\pi}\int_a^b \frac{\psi(y)}{x-y}\,dy = \frac{1}{\pi x}\int_a^b \frac{\psi(y)}{1-y/x}\,dy = \frac{1}{\pi x}\int_a^b \psi(y)\sum_{n=0}^{\infty}\Big(\frac{y}{x}\Big)^n\,dy.$$

Now for sufficiently large x; $\Big(|x| > \max(|a|,|b|)\Big)$, we end up with

$$\Psi(x) = \frac{1}{\pi x}\sum_{n=0}^{\infty} x^{-n}\int_a^b \psi(y)y^n\,dy \frac{1}{\pi x}\sum_{n=N}^{\infty} x^{-n}\int_a^b \psi(y)y^n\,dy \approx \frac{1}{\pi}x^{-N-1}\int_a^b \psi(y)y^{N+1}\,dy.$$