1

Linjärt heltalsproblem

$$z^* = \min \; x_1 + 2x_2 \qquad x^* = (1,0),\; z^* = 1$$
$$\text{då} \; 2x_1 + 2x_2 \geq 1$$
$$x_1, x_2 \in \{0,1\},$$

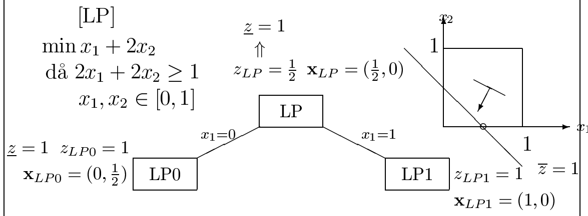
$$z^*_{LP} = \min \; x_1 + 2x_2 \qquad x^*_{LP} = \left(\frac{1}{2}, 0\right),\; z^*_{LP} = \frac{1}{2}$$
$$\text{då} \; 2x_1 + 2x_2 \geq 1$$
$$x_1, x_2 \in [0,1]$$

$$z^*_{LP} \leq z^*$$

2

Lite om Branch–and–Bound

[LP]
$$\min x_1 + 2x_2$$
$$\text{då } 2x_1 + 2x_2 \geq 1$$
$$x_1, x_2 \in [0,1]$$

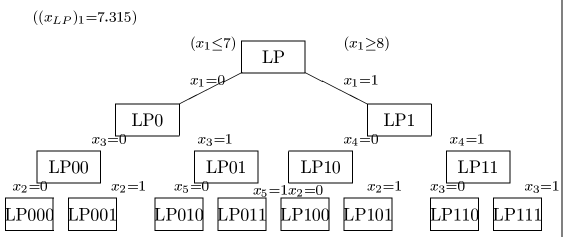


[LP0]
$$\min 2x_2$$
$$\text{då } 2x_2 \geq 1$$
$$x_2 \in [0,1]$$

[LP1]
$$\min 1 + 2x_2$$
$$\text{då } 2 + 2x_2 \geq 1$$
$$x_2 \in [0,1]$$

3

Ett Branch–and–Bound-träd



o.s.v.

4

Branch–and–price för linjära 0/1-problem

$$[\text{IP}] \qquad z^*_{\text{IP}} = \min \; \mathbf{c}^T\mathbf{x}$$
$$\text{då} \; \mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \in X = \{\mathbf{x} \in B^n \mid \mathbf{D}\mathbf{x} = \mathbf{d}\} = \{\bar{\mathbf{x}}^p \mid p \in \mathcal{P}\}$$

Inre representation (och konvexifiering):

$$\texttt{conv}\; X = \left\{ \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p \,\middle|\, \sum_{p \in \mathcal{P}} \lambda_p = 1;\; \lambda_p \geq 0,\; p \in \mathcal{P} \right\}$$

Låt $c_p = \mathbf{c}^T\bar{\mathbf{x}}^p$ och $\mathbf{a}_p = \mathbf{A}\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$.

5

Starkare formulering—Masterproblem

$$z^*_{\text{IP}} = z^*_{\text{CP}} = \min \sum_{p \in \mathcal{P}} c_p \lambda_p$$
$$[\text{CP}] \qquad \text{då} \sum_{p \in \mathcal{P}} \mathbf{a}_p \lambda_p = \mathbf{b}$$
$$\sum_{p \in \mathcal{P}} \lambda_p = 1$$
$$\lambda_p \in \{0,1\}, \qquad p \in \mathcal{P}$$

- En kontinuerlig relaxering ([CP^{cont}], till $\lambda_p \geq 0$) av [CP] ger samma undre gräns som den Lagrange-dual som erhålls vid relaxering av bivillkoren $\mathbf{A}\mathbf{x} = \mathbf{b}$ i [IP]. ($z^*_{LP} \leq z^{cont}_{CP} \leq z^*_{CP}$)
- En kontinuerlig relaxering [LP] av [IP] ger en undre gräns som aldrig kan vara bättre än någon Lagrangedual för detsamma.

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Begränsat master-problem

Låt $\bar{\mathcal{P}} \subseteq \mathcal{P}$, d.v.s., endast en delmängd av kolumnerna är genererade.

$$z^*_{\text{CP}} \geq z^{cont}_{\text{CP}} \leq \bar{z}_{\text{CP}} = \min \sum_{p \in \bar{\mathcal{P}}} c_p \lambda_p$$
$$[\overline{\text{CP}}] \qquad \text{då} \sum_{p \in \bar{\mathcal{P}}} \mathbf{a}_p \lambda_p = \mathbf{b}$$
$$\sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 \qquad (*)$$
$$\lambda_p \geq 0, \qquad p \in \bar{\mathcal{P}}$$

Generera kolumner $\begin{pmatrix} c_p \\ \mathbf{a}_p \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{c}^T\bar{\mathbf{x}}^p \\ \mathbf{A}\bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ tills en (nästan) optimal lösning till [CP^{cont}], $\widehat{\lambda}_p$, $p \in \bar{\mathcal{P}}$, är funnen $\Rightarrow \widehat{\mathbf{x}} = \sum_{p \in \bar{\mathcal{P}}} \widehat{\lambda}_p \bar{\mathbf{x}}^p$

7

Om $0 < \hat{x}_j < 1$ för något j

Förgrena över variabeln x_j (två grenar i B&B-trädet):

$$x_j = 0 \qquad \text{eller} \qquad x_j = 1$$
$$\Updownarrow \qquad\qquad \Updownarrow$$
$$x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}^p_j = 0 \qquad x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}^p_j = 1$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$\text{kol. bort} \quad \sum_{p \in \bar{\mathcal{P}}: \bar{x}^p_j = 1} \lambda_p = 0 \qquad \sum_{p \in \bar{\mathcal{P}}: \bar{x}^p_j = 1} \lambda_p = 1 \quad \text{i.st.f. } (*)$$
$$\Updownarrow \qquad\qquad \Updownarrow$$
$$\text{i.st.f. } (*) \quad \sum_{p \in \bar{\mathcal{P}}: \bar{x}^p_j = 0} \lambda_p = 1 \qquad \sum_{p \in \bar{\mathcal{P}}: \bar{x}^p_j = 0} \lambda_p = 0 \quad \text{kol. bort}$$

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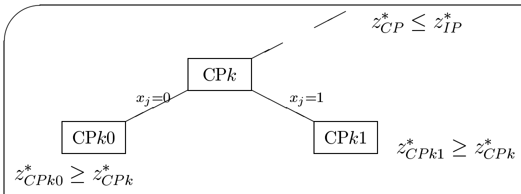


- I varje nod (CP, CP0, CP1, ...): Generera kolumner tills (nästan) optimalt (alla reduc. kostn. ≥ 0) eller verifierat otillåtet
- Om $\mathbf{x}^*_{CPk\ell\ldots}$ tillåten $\Rightarrow z^*_{CPk\ell\ldots} \geq z^*_{IP} \Rightarrow$ Kapa grenen $(k, \ell, \ldots)$

$\Rightarrow$ Kapa grenar $(r, s, \ldots)$ med $z^*_{CPrs\ldots} \geq z^*_{CPk\ell\ldots}$

- Kapa grenar med tom tillåten mängd

Kolumngenererings-subproblemet, reducerade kostnader

- $\min_{\mathbf{x}\in X^k}(\mathbf{c}-\mathbf{A}^T\widehat{\boldsymbol{\pi}}^k)^T\mathbf{x}-\widehat{q}^k \stackrel{\text{def.}}{=} (\mathbf{c}-\mathbf{A}^T\widehat{\boldsymbol{\pi}}^k)^T\bar{\mathbf{x}}^p-\widehat{q}^k \stackrel{\text{def.}}{=} \bar{c}(\bar{\mathbf{x}}^p)$
- $(\widehat{\boldsymbol{\pi}}^k,\widehat{q}^k)$ är duallösning till det begränsade master-problemet och $X^k = X\cap\{\mathbf{x}\mid x_j = k\}$, $k\in\{0,1\}$ (o.s.v. ner i trädet)
- Om $\bar{c}(\bar{\mathbf{x}}^p)<0$ så är $\begin{pmatrix}\mathbf{c}^T\bar{\mathbf{x}}^p\\ \mathbf{A}\bar{\mathbf{x}}^p\\ 1\end{pmatrix}$ en ny kolumn i [CPk]
- Minimering? Det räcker med $\bar{\mathbf{x}}^r$ sådan att $\bar{c}(\bar{\mathbf{x}}^r)<0$
- Om $\bar{c}(\bar{\mathbf{x}}^p)\ge 0$ så behövs inga fler kolumner för att lösa [CPk] till optimalitet.
- Samma kolumner kan genereras i olika noder i B&B-trädet $\Rightarrow$ skapa en "kolumn-pool" att testa m.a.p. reducerade kostnader

Ett exempel löst med Branch&price

$$\begin{array}{llll} z^*_{IP} = \min & x_1+2x_2 & = z^*_{CP} \ge z^{cont}_{CP} = z^*_{LP} = \min & x_1+2x_2\\ \text{då} & 2x_1+2x_2\ge 1 & \text{då} & 2x_1+2x_2\ge 1\\ & x_1,x_2\in\{0,1\} & & 0\le x_1,x_2\le 1\end{array}$$

$$\texttt{conv}X = \texttt{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\} = \left\{\begin{pmatrix}\lambda_3+\lambda_4\\ \lambda_2+\lambda_4\end{pmatrix}\middle|\sum_{p=1}^4\lambda_p = 1;\ \begin{array}{r}\lambda_p\ge 0\\ \forall p\end{array}\right\}$$

$$[\text{CP}]\qquad \begin{array}{lrcl} z^{cont}_{CP} = \min & 2\lambda_2+\lambda_3+3\lambda_4 & & \\ \text{då} & 2\lambda_2+2\lambda_3+4\lambda_4 & \ge & 1\\ & \lambda_1+\lambda_2+\lambda_3+\lambda_4 & = & 1\\ & \lambda_1,\lambda_2,\lambda_3,\lambda_4 & \ge & 0\end{array}$$

Startkolumner: λ_1 och λ_3

Välj t.ex. $\begin{pmatrix}0\\0\end{pmatrix}$ och $\begin{pmatrix}1\\0\end{pmatrix}$, d.v.s. variablerna λ_1 och λ_3

$$\begin{array}{lrcl} z^{cont}_{CP}\le\min & \lambda_3 & & \\ \text{då} & 2\lambda_3 & \ge & 1\\ & \lambda_1+\lambda_3 & = & 1\\ & \lambda_1,\lambda_3 & \ge & 0\end{array} \quad = \quad \begin{array}{lrcl}\max & \pi+q & & \\ \text{då} & q & \le & 0\\ & 2\pi+q & \le & 1\\ & \pi & \ge & 0\end{array}$$

Lösning: $(\widehat{\lambda}_1,\widehat{\lambda}_3) = (\frac12,\frac12)\Rightarrow\widehat{\mathbf{x}} = (\frac12,0)$, $\widehat{\pi}=\frac12$, $\widehat{q}=0$ Reducerade kostnader: $\min_{\mathbf{x}\in[0,1]^2}(0,1)\mathbf{x} = 0\Rightarrow$ Optimum för CP!

$$\begin{array}{lccc}\text{Fixeringar:} & x_1 = 0 & \text{eller} & x_1 = 1\\ & \Downarrow & & \Downarrow\\ & \lambda_3 = 0 & & \lambda_1 = 0\end{array}$$

Förgrening, vänster (CP0): $\lambda_3 = 0$

$$\begin{array}{lrcl}\min & 0 & & \\ \text{då} & 0 & \ge & 1\\ & \lambda_1 & = & 1\\ & \lambda_1 & \ge & 0\end{array} \Rightarrow \left[\begin{array}{c}\text{otillåtet}\\ \Downarrow\\ \text{lägg till}\\ \text{kolumn}\end{array}\right] \Rightarrow \begin{array}{lrcl} z_{CP0}\le\min & 2\lambda_2 & & \\ \text{då} & 2\lambda_2 & \ge & 1\\ & \lambda_1+\lambda_2 & = & 1\\ & \lambda_1,\lambda_2 & \ge & 0\end{array}$$

$$= \begin{array}{lrcl}\max & \pi+q & & \\ \text{då} & q & \le & 0\\ & 2\pi+q & \le & 2\\ & \pi & \ge & 0\end{array}$$

Lösning: $(\widehat{\lambda}_1,\widehat{\lambda}_2) = (\frac12,\frac12)$
$\Rightarrow\widehat{\mathbf{x}} = (0,\frac12)$
$\widehat{\pi} = 1,\quad \widehat{q} = 0$

Reducerade kostnader: $\min_{\mathbf{x}\in[0,1]^2}(-1,0)\mathbf{x}-0 = -1<0$
$\Rightarrow$ Ny kolumn! (λ_3 eller λ_4, men $\lambda_3\equiv 0$) $\Rightarrow$ Välj λ_4

$$\begin{array}{lrcl} z_{CP0}\le\min & 2\lambda_2+3\lambda_4 & & \\ \text{då} & 2\lambda_2+4\lambda_4 & \ge & 1\\ & \lambda_1+\lambda_2+\lambda_4 & = & 1\\ & \lambda_1,\lambda_2,\lambda_4 & \ge & 0\end{array} \quad = \quad \begin{array}{lrcl}\max & \pi+q & & \\ \text{då} & q & \le & 0\\ & 2\pi+q & \le & 2\\ & 4\pi+q & \le & 3\\ & \pi & \ge & 0\end{array}$$

- Lösning: $(\widehat{\lambda}_1,\widehat{\lambda}_3,\widehat{\lambda}_4) = (\frac34,0,\frac14)\Rightarrow\widehat{\mathbf{x}} = (\frac14,\frac14)$, $\widehat{\pi}=\frac34$, $\widehat{q}=0$
- Reducerade kostnader: $\min_{\mathbf{x}\in[0,1]^2}(-\frac12,\frac12)\mathbf{x} = -\frac12\Rightarrow$
- Generera ny kolumn: λ_3, men $\lambda_3\equiv 0\Rightarrow$ Optimum för CP0

Förgrening, höger (CP1): $\lambda_1 = 0$

$$\begin{array}{lrcl} z_{CP1}\le\min & \lambda_3 & & \\ \text{då} & 2\lambda_3 & \ge & 1\\ & \lambda_3 & = & 1\\ & \lambda_3 & \ge & 0\end{array} \quad = \quad \begin{array}{lrcl}\max & \pi+q & & \\ \text{då} & 2\pi+q & \le & 1\\ & \pi & \ge & 0\end{array}$$

- Lösning: $\widehat{\lambda}_3 = 1\Rightarrow\widehat{\mathbf{x}} = (1,0)$, $\widehat{\pi}=0$, $\widehat{q}=1$
- Reducerade kostnader: $\min_{\mathbf{x}\in[0,1]^2}(1,2)\mathbf{x}-1 = -1<0\Rightarrow$
- Generera ny kolumn: λ_1, men $\lambda_1\equiv 0\Rightarrow$ Optimum för CP1 !!

Förgrening, vänster, vänster: (CP00) $\lambda_2 = \lambda_4 = 0$

CP00: $\lambda_2 = \lambda_3 = \lambda_4 = 0\Rightarrow$ otillåtet

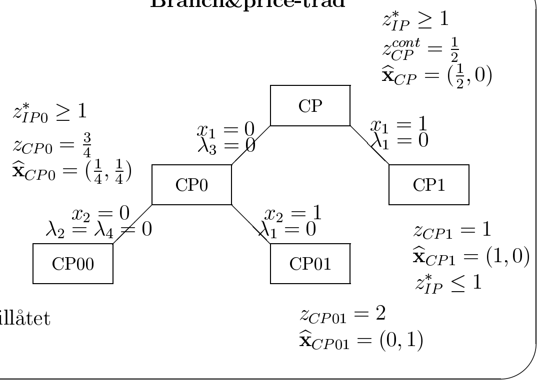
Förgrening, vänster, höger: (CP01) $\lambda_1 = 0$

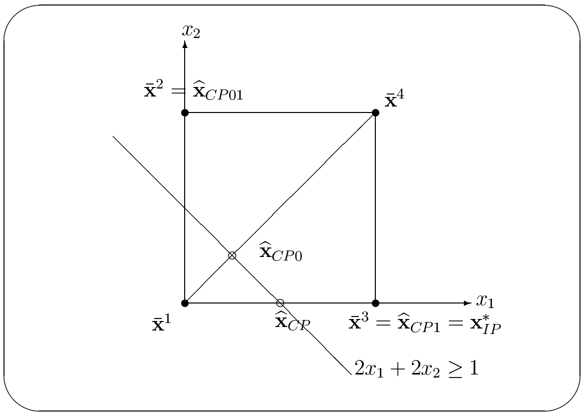
CP01: $\lambda_1 = \lambda_3 = 0$

$$\begin{array}{lrcl} z_{CP01}\le\min & 2\lambda_2+3\lambda_4 & & \\ \text{då} & 2\lambda_2+4\lambda_4 & \ge & 1\\ & \lambda_2+\lambda_4 & = & 1\\ & \lambda_2,\lambda_4 & \ge & 0\end{array} \quad = \quad \begin{array}{lrcl}\max & \pi+q & & \\ \text{då} & 2\pi+q & \le & 2\\ & 4\pi+q & \le & 3\\ & \pi & \ge & 0\end{array}$$

- Lösning: $(\widehat{\lambda}_2,\widehat{\lambda}_4) = (1,0)\Rightarrow\widehat{\mathbf{x}} = (0,1)$, $\widehat{\pi}=0$, $\widehat{q}=2$
- Reducerade kostnader: $\min_{\mathbf{x}\in[0,1]^2}(1,2)\mathbf{x}-2 = -2<0$
  $\Rightarrow$ Generera ny kolumn: λ_1, men $\lambda_1\equiv 0$
  $\Rightarrow$ Generera ny kolumn: λ_3, men $\lambda_3\equiv 0$
  $\Rightarrow$ Optimum för CP01 !!

Branch&price-träd




x_2
$\bar{\mathbf{x}}^2 = \widehat{\mathbf{x}}_{CP01}$
$\bar{\mathbf{x}}^4$
$\widehat{\mathbf{x}}_{CP0}$
x_1
$\bar{\mathbf{x}}^1$
$\widehat{\mathbf{x}}_{CP}$
$\bar{\mathbf{x}}^3 = \widehat{\mathbf{x}}_{CP1} = \mathbf{x}^*_{IP}$
$2x_1 + 2x_2 \geq 1$