Linear integer problem

$$
\begin{aligned}
z^* = \min \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & 2x_1 + 2x_2 \geq 1 \\
& x_1, x_2 \in \{0, 1\},
\end{aligned}
\qquad x^* = (1, 0),\ \ z^* = 1
$$

$$
\begin{aligned}
z^*_{LP} = \min \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & 2x_1 + 2x_2 \geq 1 \\
& x_1, x_2 \in [0, 1]
\end{aligned}
\qquad x^*_{LP} = \left(\frac{1}{2}, 0\right),\ \ z^*_{LP} = \frac{1}{2}
$$

$$z^*_{LP} \leq z^*$$

About Branch–and–Bound

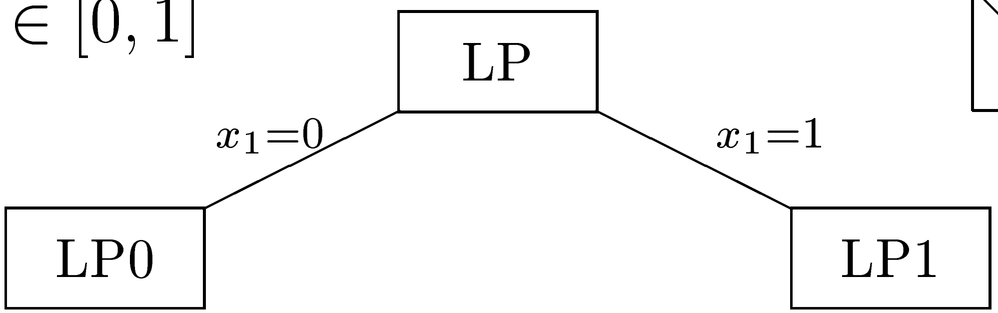
[LP]

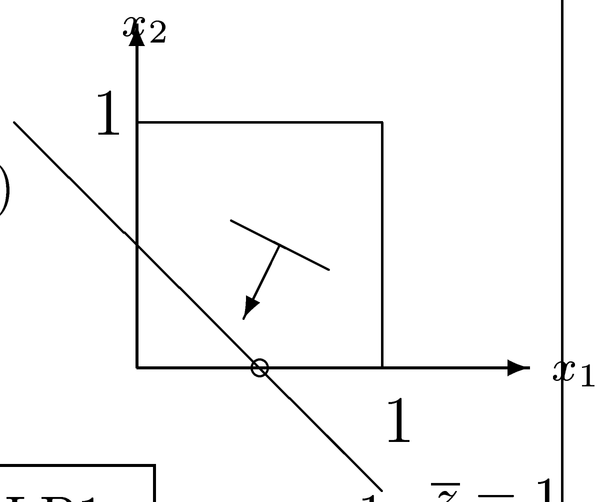
$$\begin{aligned} &\min x_1 + 2x_2 \\ &\text{s.t. } 2x_1 + 2x_2 \geq 1 \\ &\quad x_1, x_2 \in [0,1] \end{aligned}$$

$\underline{z} = 1$

$\Uparrow$

$z_{LP} = \frac{1}{2}$ $\mathbf{x}_{LP} = (\frac{1}{2}, 0)$





$\underline{z} = 1$ $z_{LP0} = 1$

$\mathbf{x}_{LP0} = (0, \frac{1}{2})$

$z_{LP1} = 1$ $\overline{z} = 1$

$\mathbf{x}_{LP1} = (1, 0)$

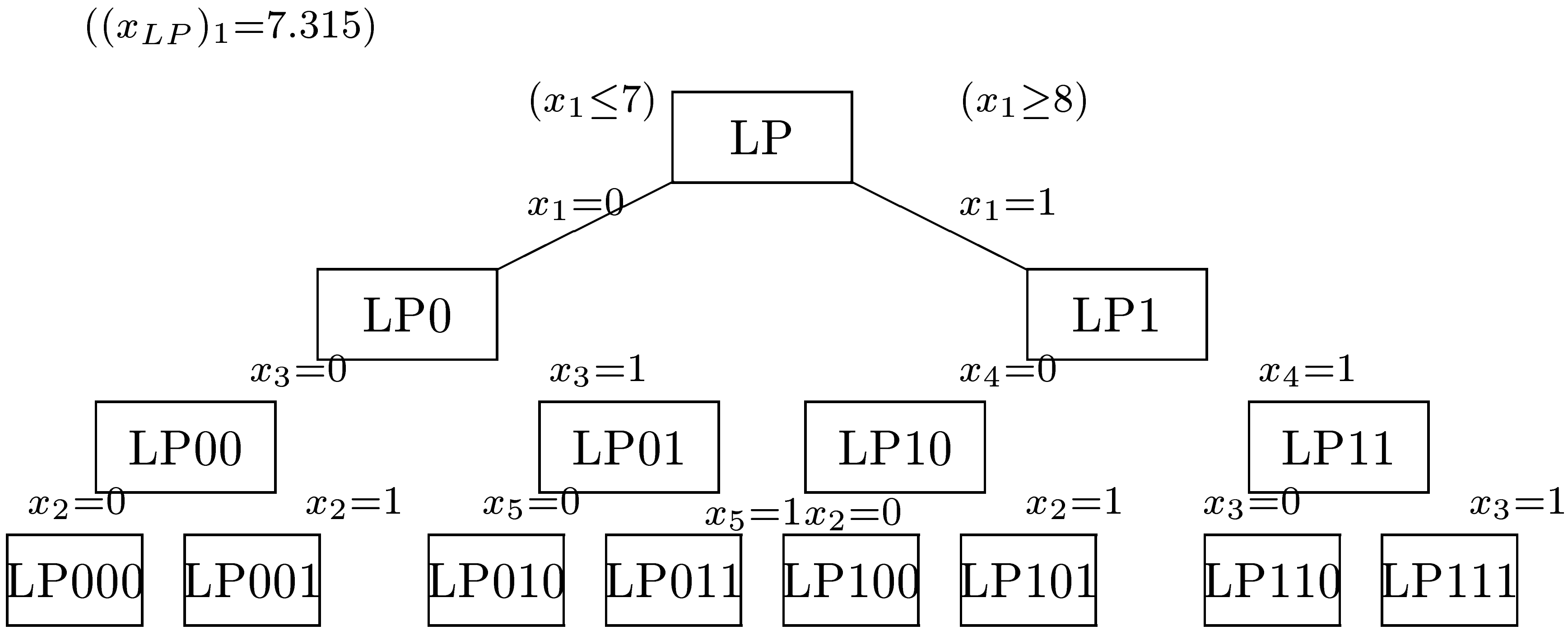
[LP0]

$$\begin{aligned} &\min 2x_2 \\ &\text{s.t. } 2x_2 \geq 1 \\ &\quad x_2 \in [0,1] \end{aligned}$$

[LP1]

$$\begin{aligned} &\min 1 + 2x_2 \\ &\text{s.t. } 2 + 2x_2 \geq 1 \\ &\quad x_2 \in [0,1] \end{aligned}$$

A Branch–and–Bound tree



etc.

Branch–and–price for linea $0/1$ problems

$$
\text{[IP]} \qquad z^*_{\mathrm{IP}} = \min \quad \mathbf{c}^T\mathbf{x}
$$
$$
\begin{aligned}
\text{s.t.} \quad \mathbf{Ax} &= \mathbf{b} \\
\mathbf{x} &\in X = \{\mathbf{x} \in B^n \mid \mathbf{Dx} = \mathbf{d}\} = \{\bar{\mathbf{x}}^p \mid p \in \mathcal{P}\}
\end{aligned}
$$

Inner representation (and convexification):

$$
\texttt{conv}\ X = \left\{ \mathbf{x} = \sum_{p\in\mathcal{P}} \lambda_p \bar{\mathbf{x}}^p \,\middle|\, \sum_{p\in\mathcal{P}} \lambda_p = 1;\ \lambda_p \geq 0,\ p \in \mathcal{P} \right\}
$$

Let $c_p = \mathbf{c}^T\bar{\mathbf{x}}^p$ and $\mathbf{a}_p = \mathbf{A}\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$.

Stronger formulation—Master problem

$$
\begin{array}{llllll}
 & z^*_{\mathrm{IP}} = z^*_{\mathrm{CP}} = & \min & \displaystyle\sum_{p\in\mathcal{P}} c_p\lambda_p & & \\
[\mathrm{CP}] & & \text{s.t.} & \displaystyle\sum_{p\in\mathcal{P}} \mathbf{a}_p\lambda_p & = & \mathbf{b} \\
 & & & \displaystyle\sum_{p\in\mathcal{P}} \lambda_p & = & 1 \\
 & & & \lambda_p & \in & \{0,1\}, \qquad p\in\mathcal{P}
\end{array}
$$

- A continuous relaxation ($[\mathrm{CP}^{cont}]$, to $\lambda_p \geq 0$) of [CP] gives the same lower bound as the Lagrange dual that corresponds to the relaxation of the constraints $\mathbf{Ax} = \mathbf{b}$ of [IP]. ($z^*_{LP} \leq z^{cont}_{CP} \leq z^*_{CP}$)
- A continuous relaxation [LP] of [IP] gives a lower bound that cannot be better than any Lagrange dual bound of [IP].

Restricted master problem

Let $\bar{\mathcal{P}} \subseteq \mathcal{P}$, i.e., only a subset of the columns are generated.

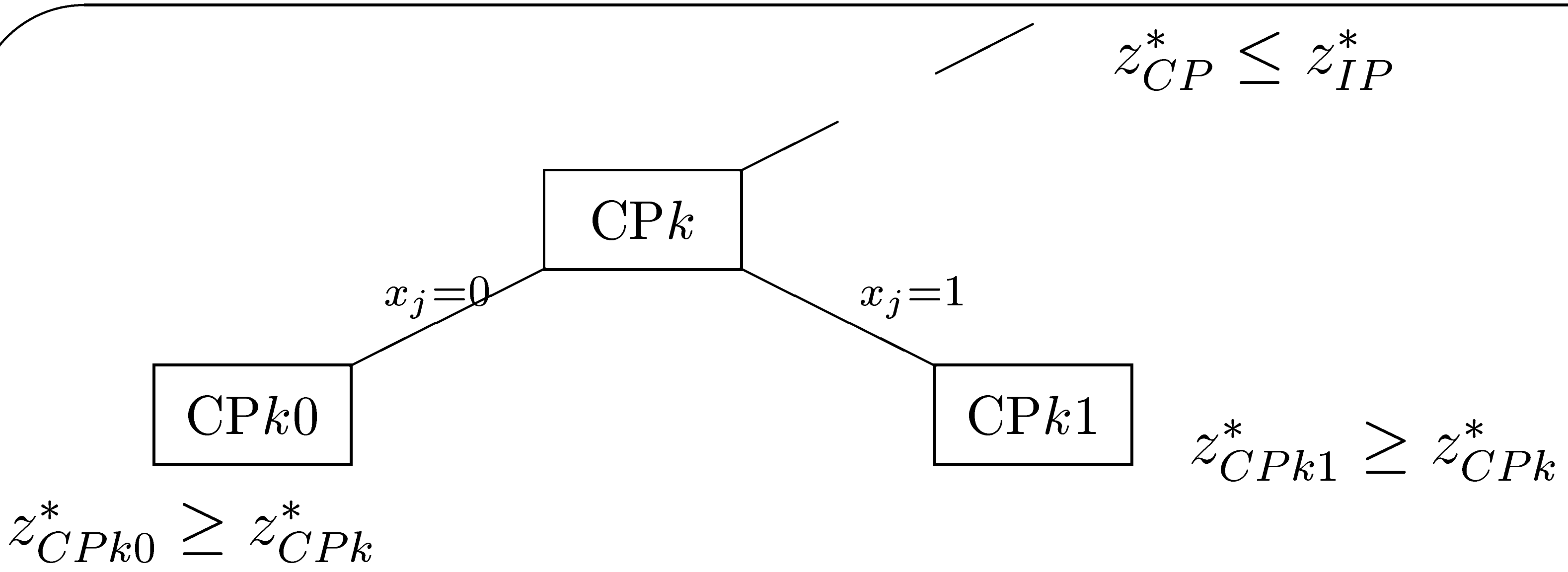
$$\begin{array}{rcl} z^*_{\mathrm{CP}} \;\geq\; z^{cont}_{\mathrm{CP}} \;\leq\; \bar{z}_{\mathrm{CP}} \;=\; \min & \displaystyle\sum_{p \in \bar{\mathcal{P}}} c_p \lambda_p & \\ [\overline{\mathrm{CP}}] \qquad\qquad \text{s.t.} & \displaystyle\sum_{p \in \bar{\mathcal{P}}} \mathbf{a}_p \lambda_p = \mathbf{b} & \\ & \displaystyle\sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 & (*) \\ & \lambda_p \geq 0, & p \in \bar{\mathcal{P}} \end{array}$$

Generate columns $\begin{pmatrix} c_p \\ \mathbf{a}_p \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^p \\ \mathbf{A}\bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ until an (almost) optimal solution to [CP^{cont}], $\widehat{\lambda}_p$, $p \in \bar{\mathcal{P}}$, is found $\Rightarrow \widehat{\mathbf{x}} = \sum_{p \in \bar{\mathcal{P}}} \widehat{\lambda}_p \bar{\mathbf{x}}^p$

If $0 < \hat{x}_j < 1$ for some j

Branch over the variabel x_j (two branches in the B&B-tree):

$$
\begin{array}{lcccl}
 & x_j = 0 & \text{or} & x_j = 1 & \\
 & \Updownarrow & & \Updownarrow & \\
 & x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 0 & & x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 1 & \\
 & \Downarrow & & \Downarrow & \\
\text{delete col's} & \sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 1} \lambda_p = 0 & & \sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 1} \lambda_p = 1 & \text{instead of } (*) \\
 & \Updownarrow & & \Updownarrow & \\
\text{instead of } (*) & \sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 0} \lambda_p = 1 & & \sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 0} \lambda_p = 0 & \text{delete col's}
\end{array}
$$



- In each node (CP, CP0, CP1, ...): Generate columns until (almost) optimal (all reduced costs ≥ 0) or verified infeasible
- If $\mathbf{x}^*_{CPk\ell\ldots}$ feasible $\Rightarrow z^*_{CPk\ell\ldots} \geq z^*_{IP} \Rightarrow$ Cut off the branch $(k, \ell, \ldots)$

$\Rightarrow$ Cut branches $(r, s, \ldots)$ with $z^*_{CPrs\ldots} \geq z^*_{CPk\ell\ldots}$

- Cut branches with empty feasible set

The column generation subproblem, reduced costs

- $\min_{\mathbf{x}\in X^k}(\mathbf{c}-\mathbf{A}^T\widehat{\boldsymbol{\pi}}^k)^T\mathbf{x}-\widehat{q}^k \stackrel{\text{def.}}{=} (\mathbf{c}-\mathbf{A}^T\widehat{\boldsymbol{\pi}}^k)^T\bar{\mathbf{x}}^p-\widehat{q}^k \stackrel{\text{def.}}{=} \bar{c}(\bar{\mathbf{x}}^p)$
- $(\widehat{\boldsymbol{\pi}}^k, \widehat{q}^k)$ is a dual solution to the restricted master problem and $X^k = X \cap \{\mathbf{x} \mid x_j = k\}$, $k \in \{0,1\}$ (etc. down the tree)
- If $\bar{c}(\bar{\mathbf{x}}^p) < 0$ then $\begin{pmatrix} \mathbf{c}^T\bar{\mathbf{x}}^p \\ \mathbf{A}\bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ is a new column in [CPk]
- Minimization? It is enough with a $\bar{\mathbf{x}}^r$ such that $\bar{c}(\bar{\mathbf{x}}^r) < 0$
- If $\bar{c}(\bar{\mathbf{x}}^p) \geq 0$ then no more columns are needed to solve [CPk] to optimality.
- Equivalent columns may be generated in different nodes in the B&B-tree $\Rightarrow$ create a "column pool" to check w.r.t. reduced costs

An instance solved by Branch&price

$$
\begin{array}{rlcll}
z^*_{IP} = \min & x_1 + 2x_2 & = z^*_{CP} \geq z^{cont}_{CP} = z^*_{LP} = & \min & x_1 + 2x_2 \\
\text{s.t.} & 2x_1 + 2x_2 \geq 1 & & \text{s.t.} & 2x_1 + 2x_2 \geq 1 \\
& x_1, x_2 \in \{0,1\} & & & 0 \leq x_1, x_2 \leq 1
\end{array}
$$

$$
\mathtt{conv}X = \mathtt{conv}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \lambda_3 + \lambda_4 \\ \lambda_2 + \lambda_4 \end{pmatrix} \middle| \sum_{p=1}^{4} \lambda_p = 1; \quad \begin{array}{r} \lambda_p \geq 0 \\ \forall p \end{array} \right\}
$$

$$
\begin{array}{lrrcl}
[\text{CP}] \qquad z^{cont}_{CP} = & \min & 2\lambda_2 + \lambda_3 + 3\lambda_4 & & \\
& \text{s.t.} & 2\lambda_2 + 2\lambda_3 + 4\lambda_4 & \geq & 1 \\
& & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & = & 1 \\
& & \lambda_1, \lambda_2, \lambda_3, \lambda_4 & \geq & 0
\end{array}
$$

Start columns: λ_1 and λ_3

Choose e.g., $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, i.e. the variables λ_1 and λ_3

$$
\begin{array}{rrcl}
z_{CP}^{cont} \leq \min & \lambda_3 & & \\
\text{s.t.} & 2\lambda_3 & \geq & 1 \\
& \lambda_1 + \lambda_3 & = & 1 \\
& \lambda_1, \lambda_3 & \geq & 0
\end{array}
\quad = \quad
\begin{array}{rrcl}
\max & \pi + q & & \\
\text{s.t.} & q & \leq & 0 \\
& 2\pi + q & \leq & 1 \\
& \pi & \geq & 0
\end{array}
$$

Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_3) = (\frac{1}{2}, \frac{1}{2}) \Rightarrow \widehat{\mathbf{x}} = (\frac{1}{2}, 0)$, $\widehat{\pi} = \frac{1}{2}$, $\widehat{q} = 0$

Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} (0,1)\mathbf{x} = 0 \Rightarrow$ Optimum for CP!

$$
\begin{array}{lccc}
\text{Fixations:} & x_1 = 0 & \text{or} & x_1 = 1 \\
& \Downarrow & & \Downarrow \\
& \lambda_3 = 0 & & \lambda_1 = 0
\end{array}
$$

Branching, left (CP0): $\lambda_3 = 0$

$$
\begin{array}{rrcl}
\min & 0 & & \\
\text{s.t.} & 0 & \geq & 1 \\
 & \lambda_1 & = & 1 \\
 & \lambda_1 & \geq & 0
\end{array}
\Rightarrow
\begin{bmatrix} \text{infeasible} \\ \Downarrow \\ \text{add} \\ \text{column} \end{bmatrix}
\Rightarrow
\begin{array}{rrcl}
z_{CP0} \leq \min & 2\lambda_2 & & \\
\text{s.t.} & 2\lambda_2 & \geq & 1 \\
 & \lambda_1 + \lambda_2 & = & 1 \\
 & \lambda_1, \lambda_2 & \geq & 0
\end{array}
$$

$$
\begin{array}{rrrcl}
= & \max & \pi + q & & \\
 & \text{s.t.} & q & \leq & 0 \\
 & & 2\pi + q & \leq & 2 \\
 & & \pi & \geq & 0
\end{array}
$$

Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_2) = (\frac{1}{2}, \frac{1}{2})$

$\Rightarrow \widehat{\mathbf{x}} = (0, \frac{1}{2})$

$\widehat{\pi} = 1, \quad \widehat{q} = 0$

Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} (-1, 0)\mathbf{x} - 0 = -1 < 0$

$\Rightarrow$ New column! (λ_3 eller λ_4, but $\lambda_3 \equiv 0$) $\Rightarrow$ Choose λ_4

$$
z_{CP0} \le \min \; 2\lambda_2 + 3\lambda_4 \quad \text{s.t.} \quad
\begin{aligned}
2\lambda_2 + 4\lambda_4 &\ge 1 \\
\lambda_1 + \lambda_2 + \lambda_4 &= 1 \\
\lambda_1, \lambda_2, \lambda_4 &\ge 0
\end{aligned}
$$

$$
= \max \; \pi + q \quad \text{s.t.} \quad
\begin{aligned}
q &\le 0 \\
2\pi + q &\le 2 \\
4\pi + q &\le 3 \\
\pi &\ge 0
\end{aligned}
$$

- Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_3, \widehat{\lambda}_4) = (\frac{3}{4}, 0, \frac{1}{4}) \Rightarrow \widehat{\mathbf{x}} = (\frac{1}{4}, \frac{1}{4})$, $\widehat{\pi} = \frac{3}{4}$, $\widehat{q} = 0$
- Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} (-\frac{1}{2}, \frac{1}{2})\mathbf{x} = -\frac{1}{2} \Rightarrow$
- Generate new column: λ_3, men $\lambda_3 \equiv 0 \Rightarrow$ Optimum for CP0

Branching, right (CP1): $\lambda_1 = 0$

$$z_{CP1} \le \min \ \lambda_3 \quad \text{s.t.} \quad 2\lambda_3 \ge 1,\ \lambda_3 = 1,\ \lambda_3 \ge 0 \quad = \quad \max \ \pi + q \quad \text{s.t.} \quad 2\pi + q \le 1,\ \pi \ge 0$$

- Solution: $\widehat{\lambda}_3 = 1 \Rightarrow \widehat{\mathbf{x}} = (1,0)$, $\widehat{\pi} = 0$, $\widehat{q} = 1$
- Reduced costs: $\min_{\mathbf{x}\in[0,1]^2}(1,2)\mathbf{x} - 1 = -1 < 0 \Rightarrow$
- Generate new column: λ_1, men $\lambda_1 \equiv 0 \Rightarrow$ Optimum for CP1 !!

Branching, left, left: (CP00) $\lambda_2 = \lambda_4 = 0$

CP00: $\lambda_2 = \lambda_3 = \lambda_4 = 0 \Rightarrow$ infeasible

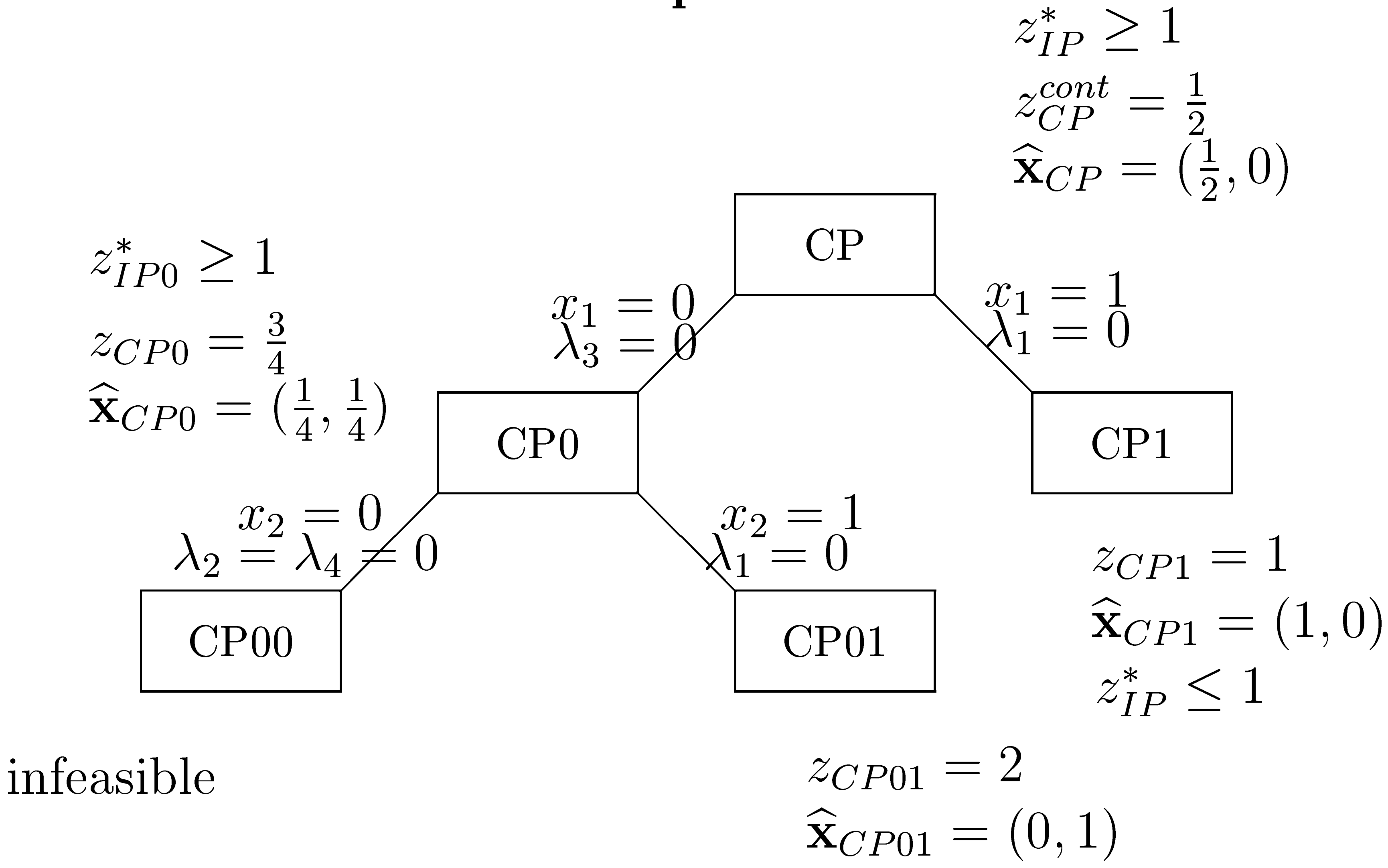
Branching, left, right: (CP01) $\lambda_1 = 0$

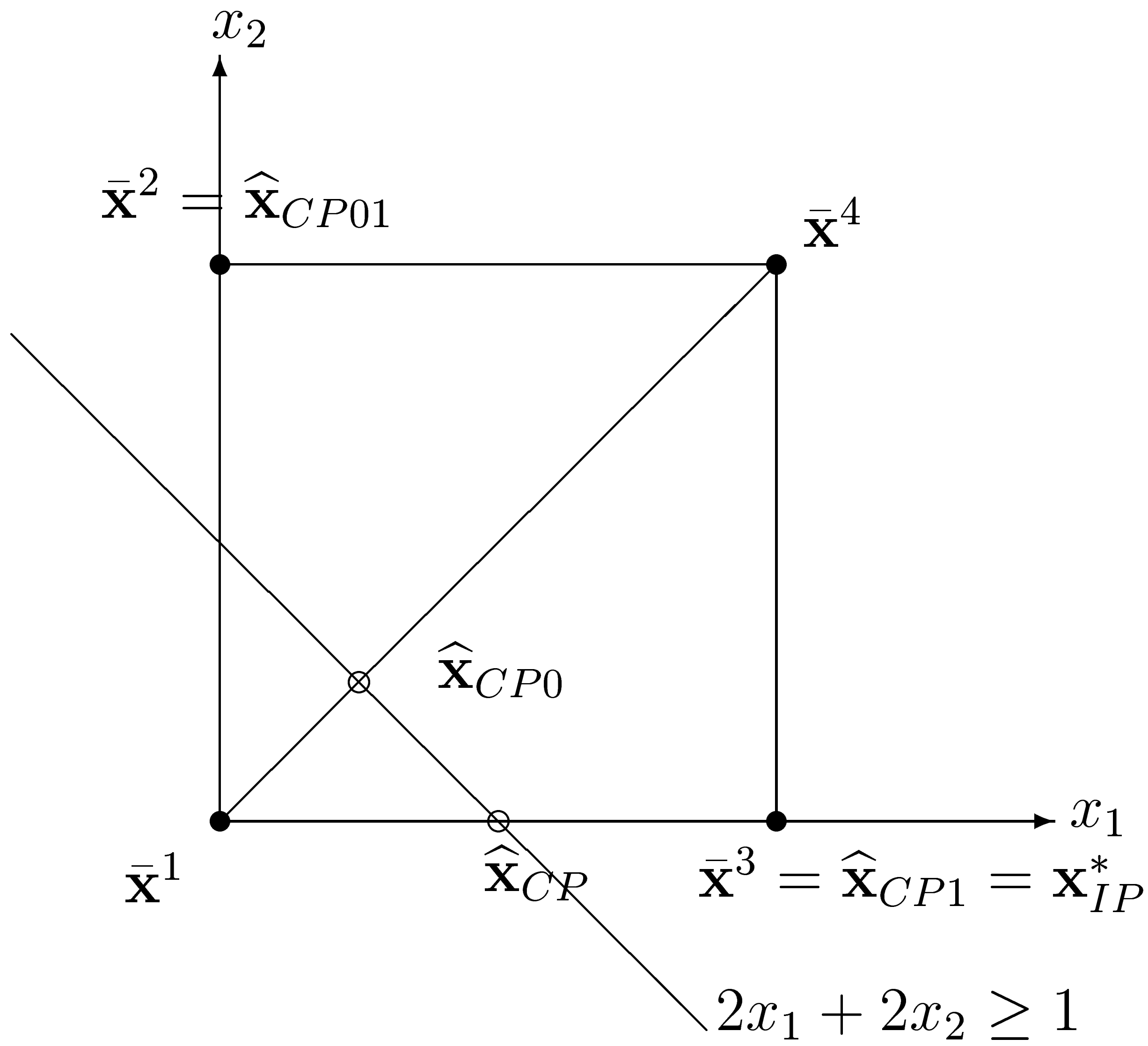
CP01: $\lambda_1 = \lambda_3 = 0$

$$
\begin{array}{rrcl}
z_{CP01} \le \min & 2\lambda_2 + 3\lambda_4 & & \\
\text{s.t.} & 2\lambda_2 + 4\lambda_4 & \ge & 1 \\
 & \lambda_2 + \lambda_4 & = & 1 \\
 & \lambda_2, \lambda_4 & \ge & 0
\end{array}
\quad = \quad
\begin{array}{rrcl}
\max & \pi + q & & \\
\text{s.t.} & 2\pi + q & \le & 2 \\
 & 4\pi + q & \le & 3 \\
 & \pi & \ge & 0
\end{array}
$$

- Solution: $(\widehat{\lambda}_2, \widehat{\lambda}_4) = (1, 0) \Rightarrow \widehat{\mathbf{x}} = (0, 1)$, $\widehat{\pi} = 0$, $\widehat{q} = 2$
- Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} (1, 2)\mathbf{x} - 2 = -2 < 0$
  $\Rightarrow$ Generate new column: λ_1, but $\lambda_1 \equiv 0$
  $\Rightarrow$ Generate new column: λ_3, but $\lambda_3 \equiv 0$
  $\Rightarrow$ Optimum for CP01 !!

Branch&price tree




x_2
$\bar{\mathbf{x}}^2 = \widehat{\mathbf{x}}_{CP01}$
$\bar{\mathbf{x}}^4$
$\widehat{\mathbf{x}}_{CP0}$
x_1
$\bar{\mathbf{x}}^1$
$\widehat{\mathbf{x}}_{CP}$
$\bar{\mathbf{x}}^3 = \widehat{\mathbf{x}}_{CP1} = \mathbf{x}^*_{IP}$
$2x_1 + 2x_2 \geq 1$