## Project course: Optimization TM, second quarter 2003

- 2 meetings per week during four weeks
- Project: selected in about two weeks
  - Models from articles or your own ideas
  - Solution methods among those discussed here
  - Present a *synopsis* at a seminar
  - Apply method to large-scale model, implement, solve, analyze
  - *Written report*: 5–10 pages excl. extras
  - *Oral presentation and opposition*
- List of literature—soon
- Separate list of projects with articles—soon
- For better grades than pass (4, 5, VG): oral exam



## Introduction

**An example of how you can utilize mathematical creativity and different optimization techniques to solve a linear, structured optimization problem with mixed 0/1- and continuous variables:**

- Lagrangian relaxation
- variable substitution
- Benders decomposition
- heuristics
- branch & bound
- greedy algorithms
- subgradient optimization



## Location av facilities that serve customers

- Potential sites: $\mathcal{J} = \{1, \ldots, n\}$ (geographical locations)
- Existing customers: $\mathcal{I} = \{1, \ldots, m\}$ (geographical locations)

$f_j =$ fixed cost of using depot $j$

$c_{ij} =$ transportation cost when customer $i$'s demand is fulfilled entirely from depot $j$

**Decision problem:**

- Which depots to open?
- Which depots to serve which customers, and how much?
- **Goal:** minimize cost
- **Assumption:** depots have unlimited capacity (to be removed)



## Decision variables:

$$y_j = \begin{cases} 1 & \text{if depot } j \text{ is set up} \\ 0 & \text{otherwise} \end{cases}$$

$x_{ij}$ = portion of customer $i$'s demand to be delivered from depot $j$

**Uncapacitated facility location (UFL)**

$$\begin{aligned}
z_0^* = \min \quad & \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} c_{ij}x_{ij} + \sum_{j\in\mathcal{J}} f_j y_j & & (0)\\
\text{s.t.} \quad & \sum_{j\in\mathcal{J}} x_{ij} = 1 & i \in \mathcal{I} \quad & (1)\\
& x_{ij} - y_j \le 0 & i \in \mathcal{I}, j \in \mathcal{J} \quad & (2)\\
& x_{ij} \in [0,1] & i \in \mathcal{I}, j \in \mathcal{J} \quad & (3)\\
& y_j \in \{0,1\} & j \in \mathcal{J} \quad & (4)
\end{aligned}$$

(0) Minimize cost

(1) Deliver precisely the demand

(2) Deliver only from open depots

(3) $x$ is the portion of the demand

(4) Do not partially open a depot

**Suppose depots have limited capacity**

$d_i =$ demand of customer $i$ ($D = \sum_{i\in\mathcal{I}} d_i$)

$b_j =$ capacity of depot $j$—if it opens

Constraints:

$$\sum_{i\in\mathcal{I}} d_i x_{ij} \le b_j y_j, \quad j \in \mathcal{J} \quad (5) \qquad (\Rightarrow x_{ij} \le y_j,\ \forall i,j)$$

$\Rightarrow$ exchange (2) for (5)



## Capacitated facility location (CFL)

$$\begin{array}{rrcll r}
z^* = \min & \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} c_{ij}x_{ij} + \sum_{j\in\mathcal{J}} f_j y_j & & & & (0)\\
\text{s.t.} & \sum_{j\in\mathcal{J}} x_{ij} & = & 1 & i\in\mathcal{I} & (1)\\
 & \sum_{i\in\mathcal{I}} d_i x_{ij} - b_j y_j & \le & 0 & j\in\mathcal{J} & (5)\\
 & x_{ij} & \in & [0,1] & i\in\mathcal{I}, j\in\mathcal{J} & (3)\\
 & y_j & \in & \{0,1\} & j\in\mathcal{J} & (4)
\end{array}$$

**Observation:** Total capacity of open depots must cover the entire demand $\Rightarrow$ an additional (redundant) constraint:

$$(1),(5) \Rightarrow \overbrace{\sum_{j\in\mathcal{J}} b_j y_j}^{capacity} \ge \sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}} d_i x_{ij} = \sum_{i\in\mathcal{I}} d_i \sum_{j\in\mathcal{J}} x_{ij} = \sum_{i\in\mathcal{I}} d_i \cdot 1 = \overbrace{D}^{demand}$$



**Trick:** Exchange $x_{ij}$ for $w_{ij}$ in constraint (1) and in "half" the objective, add the constraints $x_{ij} = w_{ij}$, and let $0 \le \alpha \le 1$.

$$\begin{array}{rrcll r}
z^* = \min & \alpha\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} c_{ij}x_{ij} + (1-\alpha)\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} c_{ij}w_{ij} + \sum_{j\in\mathcal{J}} f_j y_j & & & & \\
\text{s.t.} & \sum_{j\in\mathcal{J}} w_{ij} & = & 1 & i\in\mathcal{I} & (1)\\
 & \sum_{i\in\mathcal{I}} d_i x_{ij} - b_j y_j & \le & 0 & j\in\mathcal{J} & (5)\\
 & \sum_{j\in\mathcal{J}} b_j y_j & \ge & D & & (6)\\
 & w_{ij} - x_{ij} & = & 0 & i\in\mathcal{I}, j\in\mathcal{J} & (7)\\
 & x_{ij} & \in & [0,1] & i\in\mathcal{I}, j\in\mathcal{J} & (3)\\
 & w_{ij} & \ge & 0 & i\in\mathcal{I}, j\in\mathcal{J} & (8)\\
 & y_j & \in & \{0,1\} & j\in\mathcal{J} & (4)
\end{array}$$



- Constraints (7) tie together $(x,y)$ with $w$.
- Lagrangian relax these with multipliers $\lambda_{ij}$

$\Rightarrow$ Lagrangian function

$$\mathcal{L}(x,w,y,\lambda) =$$
$$= \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\left[\alpha c_{ij}x_{ij} + (1-\alpha)c_{ij}w_{ij} + \overbrace{\lambda_{ij}(w_{ij}-x_{ij})}^{\text{penalty}}\right] + \sum_{j\in\mathcal{J}} f_j y_j$$
$$= \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}(\alpha c_{ij} - \lambda_{ij})x_{ij} + \sum_{j\in\mathcal{J}} f_j y_j + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\left[(1-\alpha)c_{ij} + \lambda_{ij}\right] w_{ij}$$

- Subproblem (for fixed value of $\lambda$):
  Minimize the Lagrangian function under constraints (1), (5), (6), (3), (8) & (4).
  Separates into one in $(x,y)$ and $|\mathcal{I}|$ in $w$

## Subproblem in $x$ and $y$:

$$\begin{array}{rrcll}
z_{xy}(\lambda) = \min\limits_{x,y} & \sum\limits_{i\in\mathcal{I}}\sum\limits_{j\in\mathcal{J}} [\alpha c_{ij} - \lambda_{ij}]\, x_{ij} & + & \sum\limits_{j\in\mathcal{J}} f_j y_j & \\
\text{s.t.} & \sum\limits_{j\in\mathcal{J}} b_j y_j & \geq & D & (6) \\
& \sum\limits_{i\in\mathcal{I}} d_i x_{ij} & \leq & b_j y_j \quad j \in \mathcal{J} & (5) \\
& x_{ij} & \in & [0,1] \quad i \in \mathcal{I}, j \in \mathcal{J} & (3) \\
& y_j & \in & \{0,1\} \quad j \in \mathcal{J} & (4)
\end{array}$$

For every $y$-solution (such that $\sum_{j\in\mathcal{J}} b_j y_j \geq D$) we have:

- If $y_j = 0$ then $x_{ij} = 0$, $i \in \mathcal{I}$
- If $y_j = 1$ then $\sum_{i\in\mathcal{I}} d_i x_{ij} \leq b_j$



## Value [in $(x,y)$-subproblem] of opening depot $j$

That is: letting $y_j = 1$ ($|\mathcal{J}|$ continuous knapsack problems)

$$[\text{KKSP}_j] \qquad v_j(\lambda) = f_j + \min \sum_{i\in\mathcal{I}} [\alpha c_{ij} - \lambda_{ij}]\, x_{ij}$$
$$\text{s.t.} \quad \sum_{i\in\mathcal{I}} d_i x_{ij} \leq b_j$$
$$x_{ij} \in [0,1] \qquad i \in \mathcal{I}$$

$\Rightarrow$ **Projection onto $y$-space** (a 0/1 knapsack problem)

$$\begin{array}{lrrcl}
[\text{0/1-KSP}] & z_{xy}(\lambda) = \min\limits_{y} & \sum\limits_{j\in\mathcal{J}} v_j(\lambda)\cdot y_j & & \\
& \text{s.t.} & \sum\limits_{j\in\mathcal{J}} b_j y_j & \geq & D \\
& & y_j & \in & \{0,1\} \quad j \in \mathcal{J}
\end{array}$$



## Solving continuous knapsack problems. [KKSP$_j$]

- Sort $\frac{\alpha c_{ij} - \lambda_{ij}}{d_i} < 0$, $i \in \mathcal{I}$, in increasing order

$\Rightarrow$ index $\{i_1, i_2, \ldots, i_m\}$, $m \leq |\mathcal{I}|$.

- If $m = 0$ then $x_{ij} = 0, i \in \mathcal{I}$. Else, let $k = 1$ and:
- Let $x_{i_k j} = \min\{1; b_j - \sum_{s=1}^{k-1} d_i x_{i_s j}\}$ and let $k = k+1$ until $\sum_{s=1}^{k} d_i x_{i_s j} = b_j$ or $k = m$.
- Solution fulfills $\sum_{i\in\mathcal{I}} d_i x_{ij} = b_j$ and $x_{ij} \in [0,1]$, $i \in \mathcal{I}$
- $v_j(\lambda) = f_j + \min \sum_{k=1}^{|\mathcal{I}|} \sum_{j\in\mathcal{J}} [\alpha c_{i_k j} - \lambda_{i_k j}]\, x_{i_k j}$

## Solving $0/1$ knapsack problems

*Not* polynomial. Solve with Branch & bound (CPLEX).

**Solution:** $y_j(\lambda) \in \{0,1\}$, $j \in \mathcal{J}$.

. $x_{ij}(\lambda) = 0$, $i \in \mathcal{I}$, if $y_j(\lambda) = 0$.

. $x_{ij}(\lambda) = x_{ij}$ by the above, $i \in \mathcal{I}$, if $y_j(\lambda) = 1$.



## Subproblem in $w$ ($|\mathcal{I}|$ semi-assignment problems):

$$[\text{AP}] \qquad z_w(\lambda) = \sum_{i\in\mathcal{I}} \left[ \begin{array}{ll} \min\limits_{w} & \sum\limits_{j\in\mathcal{J}} [(1-\alpha)c_{ij} + \lambda_{ij}]\, w_{ij} \\ \text{s.t.} & \sum\limits_{j\in\mathcal{J}} w_{ij} = 1, \quad w_{ij} \geq 0 \quad j \in \mathcal{J} \end{array} \right]$$

## Solving semi-assignment problem $i$

(special case of [KKSP]):

- Find $\ell_i$ such that $(1-\alpha)c_{i\ell_i} + \lambda_{i\ell_i} = \min\limits_{j\in\mathcal{J}}\{(1-\alpha)c_{ij} + \lambda_{ij}\}$
- Let $w_{i\ell_i}(\lambda) = 1$, $w_{ij}(\lambda) = 0$, $j \neq \ell_i$

## Value of relaxed problem for fixed value of $\lambda$

$$z(\lambda) = \underbrace{z_{xy}(\lambda)}_{\text{difficult}} + \underbrace{z_w(\lambda)}_{\text{simple}}$$

- Can show that $z(\lambda) \le z^*$ for all $\lambda \in \Re^{|\mathcal{I}|\times|\mathcal{J}|}$ (weak duality)
- $\lambda_{ij}$ is the penalty for violating $w_{ij} = x_{ij}$
- Find best underestimate of $z^*$ $\Leftrightarrow$ find "optimal" values of penalties $\lambda_{ij}$
- That is: $\max\limits_{\lambda\in\Re^{|\mathcal{I}|\times|\mathcal{J}|}} z(\lambda) \le z^*$ (most often $\max\limits_{\lambda\in\Re^{|\mathcal{I}|\times|\mathcal{J}|}} z(\lambda) < z^*$)



## How to find better value of $\lambda_{ij}$?

Penalty: $\min \ldots \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} \lambda_{ij}(w_{ij} - x_{ij})$

- If $w_{ij}(\lambda) > x_{ij}(\lambda) \Rightarrow$ Increase value of $\lambda_{ij}$ (more expensive to violate constraint)
- If $w_{ij}(\lambda) < x_{ij}(\lambda) \Rightarrow$ Decrease value of $\lambda_{ij}$ (more expensive to violate constraint)
- Iterative method (subgradient algorithm) to find optimal penalties $\lambda^*$:

$$\lambda_{ij}^{t+1} = \lambda_{ij}^t + \rho_t\left[w_{ij}(\lambda^t) - x_{ij}(\lambda^t)\right], \qquad t = 0, 1, \ldots$$

where $\rho_t > 0$ is a step length, decreasing with $t$

- Use *heuristic* from every $[x(\lambda^t), w(\lambda^t), y(\lambda^t)]$ to yield a *feasible solution* to CFL (open more depots, send only from open depots, $x = w$, ...)



## Example: $|\mathcal{I}| = 4$, $|\mathcal{J}| = 3$, $\alpha = \frac{1}{2}$

$$(c_{ij}) = \begin{bmatrix} 6 & 2 & 4 \\ 2 & 8 & 4 \\ 16 & 2 & 6 \\ 10 & 12 & 4 \end{bmatrix}, (f_j) = \begin{bmatrix} 11 \\ 16 \\ 21 \end{bmatrix}, (d_i) = \begin{bmatrix} 6 \\ 4 \\ 8 \\ 5 \end{bmatrix}, (b_j) = \begin{bmatrix} 12 \\ 10 \\ 13 \end{bmatrix}$$

$$\begin{aligned} z_{xy}(\lambda) = \min \quad & \textstyle\sum_{j=1}^3 v_j(\lambda)\cdot y_j \\ \text{s.t.} \quad & 12y_1 + 10y_2 + 13y_3 \ge 23 \\ & y \in \{0,1\}^3 \end{aligned}$$

$$\mathrm{Lt}(\lambda_{ij}^t) = \begin{bmatrix} 7 & 0 & 0 \\ 3 & 10 & 2 \\ 5 & 2 & 0 \\ 0 & 7 & 5 \end{bmatrix}$$

$$\underbrace{\ldots \Rightarrow \ldots}_{\text{(next page)}} \quad z_{xy}(\lambda) = \min \; 5y_1 + 8.875y_2 + 18y_3 \quad \text{s.t.} \; 12y_1 + 10y_2 + 13y_3 \ge 23, \quad y \in \{0,1\}^3$$



$$\begin{aligned} v_1(\lambda^t) = 11 + \quad & \min \quad -4x_{11} - 2x_{21} + 3x_{31} + 5x_{41} \\ & \text{s.t.} \quad 6x_{11} + 4x_{21} + 8x_{31} + 5x_{41} \le 12, \quad x_{\cdot 1} \in [0,1]^4 \\ \Rightarrow \quad & x_{11} = x_{21} = 1,\; x_{31} = x_{41} = 0, \quad v_1(\lambda^t) = 5 \end{aligned}$$

$$\begin{aligned} v_2(\lambda^t) = 16 + \quad & \min \quad x_{12} - 6x_{22} - x_{32} - x_{42} \\ & \text{s.t.} \quad 6x_{12} + 4x_{22} + 8x_{32} + 5x_{42} \le 10, \quad x_{\cdot 2} \in [0,1]^4 \\ \Rightarrow \quad & x_{22} = x_{42} = 1,\; x_{32} = \frac{1}{8},\; x_{12} = 0, \quad v_2(\lambda^t) = 8.875 \end{aligned}$$

$$\begin{aligned} v_3(\lambda^t) = 21 + \quad & \min \quad 2x_{13} + 0x_{23} + 3x_{33} - 3x_{43} \\ & \text{s.t.} \quad 6x_{13} + 4x_{23} + 8x_{33} + 5x_{43} \le 13, \quad x_{\cdot 3} \in [0,1]^4 \\ \Rightarrow \quad & x_{23} = x_{43} = 1,\; x_{13} = x_{33} = 0, \quad v_3(\lambda^t) = 18 \end{aligned}$$

## Solution to $(x, y)$ problem for $\lambda = \lambda^t$

$$y(\lambda^t) = (1,0,1),\ x(\lambda^t) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ z_{xy}(\lambda^t) = 5 + 0 + 18 = 23$$

## $w$ problem separates into one for each customer $i$

$z_w(\lambda^t) = \sum_{i=1}^4 z_w^i(\lambda^t)$ where $(1 - \alpha = \frac{1}{2})$

$$z_w^i(\lambda^t) = \min \quad \sum_{j=1}^{3} \left[(1-\alpha)c_{ij} + \lambda_{ij}^t\right] w_{ij}$$

$$\text{s.t.} \quad \sum_{j=1}^{3} w_{ij} = 1, \quad w_{ij} \geq 0,\ j = 1,2,3$$



$$\begin{aligned}
z_w^1(\lambda^t) = \ & \min \ 10w_{11} + w_{12} + 2w_{13} \\
& \text{s.t. } w_{11} + w_{12} + w_{13} = 1, \quad w_{1j} \geq 0,\ j = 1,2,3 \\
\Rightarrow \ & w_{12}(\lambda^t) = 1,\ w_{11}(\lambda^t) = w_{13}(\lambda^t) = 0,\ z_w^1(\lambda^t) = 1
\end{aligned}$$

$$\begin{aligned}
z_w^2(\lambda^t) = \ & \min \ 4w_{21} + 14w_{22} + 4w_{23} \\
& \text{s.t. } w_{21} + w_{22} + w_{23} = 1, \quad w_{2j} \geq 0,\ j = 1,2,3 \\
\Rightarrow \ & w_{21}(\lambda^t) = 1,\ w_{22}(\lambda^t) = w_{23}(\lambda^t) = 0,\ z_w^2(\lambda^t) = 4
\end{aligned}$$

$$\begin{aligned}
z_w^3(\lambda^t) = \ & \min \ 13w_{31} + 3w_{32} + 3w_{33} \\
& \text{s.t. } w_{31} + w_{32} + w_{33} = 1, \quad w_{3j} \geq 0,\ j = 1,2,3 \\
\Rightarrow \ & w_{32}(\lambda^t) = w_{33}(\lambda^t) = \frac{1}{2},\ w_{31}(\lambda^t) = 0,\ z_w^3(\lambda^t) = 3
\end{aligned}$$

$$\begin{aligned}
z_w^4(\lambda^t) = \ & \min \ 5w_{41} + 13w_{42} + 7w_{43} \\
& \text{s.t. } w_{41} + w_{42} + w_{43} = 1, \quad w_{4j} \geq 0,\ j = 1,2,3 \\
\Rightarrow \ & w_{41}(\lambda^t) = 1,\ w_{42}(\lambda^t) = w_{43}(\lambda^t) = 0,\ z_w^4(\lambda^t) = 5
\end{aligned}$$



## Solution to $w$ problem

$$w(\lambda^t) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix},\ z_w(\lambda^t) = 13, \quad \begin{array}{l} z(\lambda^t) = z_{xy}(\lambda^t) + z_w(\lambda^t) = 35 \\ \Rightarrow z^* \geq 35 \end{array}$$

**New $\lambda$ vector (e.g., $\rho_t = 8$):**

$$\begin{aligned}
\lambda^{t+1} &= \lambda^t + \rho_t\left[w(\lambda^t) - x(\lambda^t)\right] \\
&= \begin{bmatrix} 7-\rho_t & \rho_t & 0 \\ 3 & 10 & 2-\rho_t \\ 5 & 2+\frac{\rho_t}{2} & \frac{\rho_t}{2} \\ \rho_t & 7 & 5-\rho_t \end{bmatrix} = \begin{bmatrix} -1 & 8 & 0 \\ 3 & 10 & -6 \\ 5 & 6 & 4 \\ 8 & 7 & -3 \end{bmatrix}
\end{aligned}$$



## Feasible solution $\Leftrightarrow x(\lambda^t) = w(\lambda^t)$? No $\Rightarrow$ Heuristic

Idea: Open depots given by $y(\lambda^t) \Rightarrow y^H = y(\lambda^t) = (1,0,1)$.
Send only from open depots ($y_j^H = 0 \Rightarrow x_{ij}^H = 0, \forall i$).
Fulfill demand but do not violate capacity restrictions:

$$\text{Let } x^H = \begin{bmatrix} \frac{7}{12} & 0 & \frac{5}{12} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \Rightarrow$$

$z^H = 6 \cdot \frac{7}{12} + 4 \cdot \frac{5}{12} + 2 + 6 + 10 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} + 11 + 21 = 52 + \frac{1}{6}$
$\Rightarrow z^* \in [35, 52 + \frac{1}{6}] = [z(\lambda^t), z^H]$ (not very good interval)

- Choice of step lengths ($\rho_t$) later (subgradient optimization, convergence to an optimal value of $\lambda$)
- Feasibility heuristics can be made more or less sophisticated
- There are more ways in which to Lagrangian relax *continuous* constraints in an optimization problem
- E.g.: Lagrangian relax (1) *or* (5)
(with multipliers $\mu_i \in \Re$ resp. $\nu_j \in \Re_+$) in the original formulation (CFL)