Column generation

An LP with very many variables $c_j, x_j \in \Re$, $\mathbf{a}_j, \mathbf{b} \in \Re^m$, $m << n$

$$\begin{array}{lrcl} \min & z = \sum_{j=1}^{n} c_j x_j & & \\ \text{s.t.} & \sum_{j=1}^{n} \mathbf{a}_j x_j & = & \mathbf{b} \\ & x_j & \geq & 0, \quad j = 1, \ldots, n \end{array}$$

The matrix $(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is too large to handle.

Assume that the number of constraints is relatively small
$\Rightarrow$ a basic matrix (simplex method) is not too large ($m \times m$)

Feasible basic solutions

$B = \{m \text{ elements from the set } \{1, \ldots, n\}\}$ is a basis if the corresponding matrix $\mathbf{B} = (\mathbf{a}_j)_{j \in B}$ has an inverse, $\mathbf{B}^{-1}$

A basic solution is given by $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and $x_j = 0$, $j \notin B$

A better basic feasible solution can be found by computing reduced costs: $\bar{c}_j = c_j - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_j$ for $j \notin B$

Let $\bar{c}_s = \min\limits_{j \notin B} \bar{c}_j$

If $\bar{c}_s < 0 \Rightarrow$ a better solution is received if x_s enters the basis

If $\bar{c}_s \geq 0 \Rightarrow \mathbf{x}_B$ is an optimal basic solution

Suppose that the columns $\mathbf{a}_j$ are defined by a set $S = \{\mathbf{a}_j \mid j = 1, \dots, n\}$ being, e.g., solutions to a system of equations (extreme points, integer points, ...)

The incoming column is then chosen by solving a "subproblem":

$$\text{Let } c(\mathbf{a}_j) = c_j$$

$$\bar{c}(\mathbf{a}(B)) = \min_{\mathbf{a} \in S} \left\{ c(\mathbf{a}) - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a} \right\}$$

$\mathbf{a}(B)$ is the column having the least reduced cost w.r.t. the basis B

If $\bar{c}(\mathbf{a}(B)) < 0$ let the column $\begin{pmatrix} c(\mathbf{a}(B)) \\ \mathbf{a}(B) \end{pmatrix}$ enter the problem

Example: Cutting stock

Supply: (long) pieces of wood of length L

Demand: b_i pieces of wood of length $\ell_i < L$, $i = 1, \dots, m$

Objective: minimize the number of long pieces needed for producing the demanded pieces

Cut pattern number j contains a_{ij} pieces of length ℓ_i, $\forall i$

Feasible pattern if $\sum_{i=1}^{m} \ell_i a_{ij} \leq L$, where $a_{ij} \geq 0$ & integer $\forall i$

Variables: x_j = number of times pattern j is used

n = total number of feasible cut pattern — very large integer

$$
\begin{array}{llll}
\min & \displaystyle\sum_{j=1}^{n} x_j & & \\
\text{s.t.} & \displaystyle\sum_{j=1}^{n} a_{ij} x_j = & b_i, & i = 1, \dots, m \\
& x_j \geq & 0, \text{ (integer)} & j = 1, \dots, n
\end{array}
$$

Start solution and new columns

Start columns, trivial: m unit columns (gives lots of waste) $\Rightarrow$

$$\begin{array}{rrcll} \min & \sum_{j=1}^{m} x_j & & & \\ \text{s.t.} & x_j & = & b_j, & j = 1, \dots, m \\ & x_j & \geq & 0, & j = 1, \dots, m \end{array}$$

Generate better patterns (integer knapsack problem): $\Rightarrow$ new column

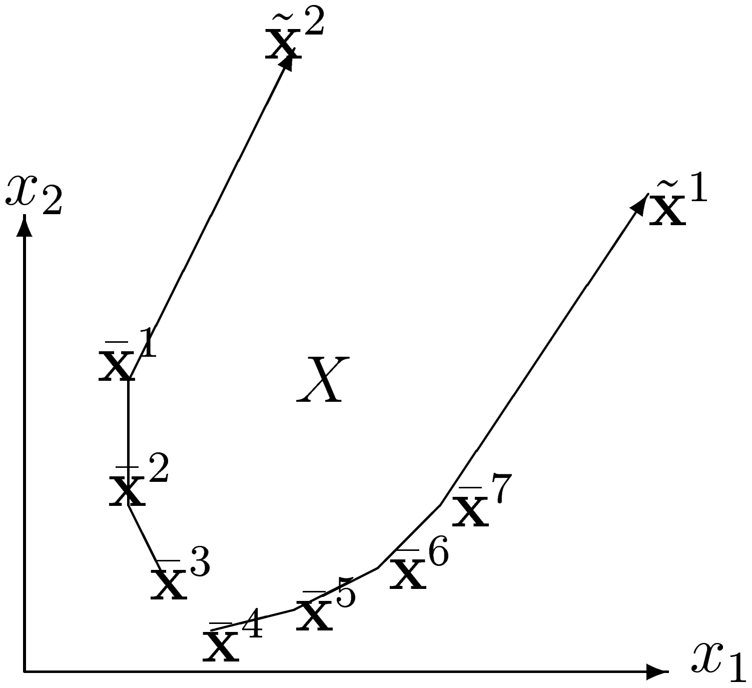
$$\left.\begin{array}{rrcll} 1 - \max\limits_{a_{ik}} & \sum_{i=1}^{m} a_{ik} & & \left[\min(c_k - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_k)\right] & \\ \text{s.t.} & \sum_{i=1}^{m} \ell_i a_{ik} & \leq & L & \\ & a_{ik} & \geq & 0, \text{ integer} & i = 1, \dots, m \end{array}\right\} \Rightarrow \text{ Solution: } \mathbf{a}_k$$

Formulation of LP on column generation form—Dantzig–Wolfe decomposition

Let $X = \{\mathbf{x} \in \Re^n_+ \mid \mathbf{D}\mathbf{x} = \mathbf{d}\}$ (or $\mathbf{D}\mathbf{x} \leq \mathbf{d}$) be a polyhedron with the **extreme points** $\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$ and the **extreme recession directions** $\tilde{\mathbf{x}}^r$, $r \in \mathcal{R}$

$\mathcal{P} = \{1, 2, \dots, 7\}$

$\mathcal{R} = \{1, 2\}$



Then

$$\mathbf{x} \in X \Longleftrightarrow \left(\begin{array}{lll} \mathbf{x} & = & \sum_{p\in\mathcal{P}} \lambda_p \bar{\mathbf{x}}^p + \sum_{r\in\mathcal{R}} \mu_r \tilde{\mathbf{x}}^r \\ & & \sum_{p\in\mathcal{P}} \lambda_p = 1 \\ & & \lambda_p \geq 0, \quad p \in \mathcal{P} \\ & & \mu_r \geq 0, \quad r \in \mathcal{R} \end{array} \right)$$

$\mathbf{x} \in X$ can be writtenas a convex combination of the extreme points plus a conical combination of the extreme directions

This *inner representation* of the set X can be used to reformulate a linear optimization problem according to the *Dantzig-Wolfe decomposition principle*, which is then solved by column generation.

$$
\begin{array}{llrcll}
\text{[LP1]} & z^* = \min & \mathbf{c}^T\mathbf{x} & & & \\
& \text{då} & \mathbf{A}\mathbf{x} & = & \mathbf{b} & \text{(complicating constraints)} \\
& & \mathbf{D}\mathbf{x} & = & \mathbf{d} & \text{(“simple” constraints)} \\
& & \mathbf{x} & \geq & \mathbf{0} &
\end{array}
$$

Let $X = \{\mathbf{x} \geq \mathbf{0} \mid \mathbf{D}\mathbf{x} = \mathbf{d}\}$ with the extreme points $\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$ and the extreme directions $\tilde{\mathbf{x}}^r$, $r \in \mathcal{R} \Rightarrow$

$$
\begin{array}{llrcll}
\text{[LP2]} & z^* = \min \displaystyle\sum_{p \in \mathcal{P}} \lambda_p(\mathbf{c}^T\bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r(\mathbf{c}^T\tilde{\mathbf{x}}^r) & & & & \\
& \text{då} \quad \displaystyle\sum_{p \in \mathcal{P}} \lambda_p(\mathbf{A}\bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r(\mathbf{A}\tilde{\mathbf{x}}^r) & = & \mathbf{b} & \mid \boldsymbol{\pi} \\
& \displaystyle\sum_{p \in \mathcal{P}} \lambda_p & = & 1 & \mid q \\
& \lambda_p, \mu_r & \geq & 0, \ p \in \mathcal{P}, r \in \mathcal{R} &
\end{array}
$$

The number of constraints in [LP2] is equal to "the number of constraints in $\mathbf{A}\mathbf{x} = \mathbf{b}$" + 1

The number of columns is very large (# extremepts./dirs. to X)

The dual of [LP2] is given by (not all extrempts./dirs. found yet)

$$
\begin{array}{llllll}
\text{[DLP2]} & z^* \leq \max\limits_{\boldsymbol{\pi},q} & \mathbf{b}^T\boldsymbol{\pi} + q & & & \\
& \text{s.t.} & (\mathbf{A}\bar{\mathbf{x}}^p)^T\boldsymbol{\pi} + q & \leq & (\mathbf{c}^T\bar{\mathbf{x}}^p), & p \in \bar{\mathcal{P}} \subseteq \mathcal{P} \quad | \lambda_p \\
& & (\mathbf{A}\tilde{\mathbf{x}}^r)^T\boldsymbol{\pi} & \leq & (\mathbf{c}^T\tilde{\mathbf{x}}^r), & r \in \bar{\mathcal{R}} \subseteq \mathcal{R} \quad | \mu_r
\end{array}
$$

with solutions $(\bar{\boldsymbol{\pi}}, \bar{q})$

Reduced cost for the variable λ_p, $p \in \mathcal{P} \setminus \bar{\mathcal{P}}$ is given by

$$(\mathbf{c}^T\bar{\mathbf{x}}^p) - (\mathbf{A}\bar{\mathbf{x}}^p)^T\bar{\boldsymbol{\pi}} - \bar{q} = (\mathbf{c} - \mathbf{A}^T\bar{\boldsymbol{\pi}})^T\bar{\mathbf{x}}^p - \bar{q}$$

Reduced cost for the variable μ_r, $r \in \mathcal{R} \setminus \bar{\mathcal{R}}$ is given by

$$(\mathbf{c}^T\tilde{\mathbf{x}}^r) - (\mathbf{A}\tilde{\mathbf{x}}^r)^T\bar{\boldsymbol{\pi}} = (\mathbf{c} - \mathbf{A}^T\bar{\boldsymbol{\pi}})^T\tilde{\mathbf{x}}^r$$

Least reduced cost is found by solving the subproblem

$$\min_{\mathbf{x}\in X} (\mathbf{c} - \mathbf{A}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} \quad \left(\text{alt:} \quad \min_{\mathbf{x}\in X} (\mathbf{c} - \mathbf{A}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} - \bar{q}\right)$$

Gives as solution an extreme point, $\bar{\mathbf{x}}^p$, or an extreme direction $\tilde{\mathbf{x}}^r$

$\Rightarrow$ a new column in [LP2]: (if < 0)

Either $\begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^p \\ \mathbf{A}\bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \mathbf{c}^T \tilde{\mathbf{x}}^r \\ \mathbf{A}\tilde{\mathbf{x}}^r \\ 0 \end{pmatrix}$

Example

$$
\begin{array}{rrrcrcrcrcrcl}
z^*_{\mathrm{IP}} & = & \min & 2x_1 & + & 3x_2 & + & x_3 & + & 4x_4 & & & \\
\text{[IP]} & & \text{s.t.} & 3x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & = & 5 & \quad | \ \mathbf{Ax} = \mathbf{b} \\
 & & & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 & \\
 & & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4} & \in & \{0, 1\} &
\end{array}
$$

$$
X = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} = \{\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^6\}
$$

Optimal solution: $\mathbf{x}^*_{\mathrm{IP}} = (0, 1, 1, 0)$ $\qquad z^*_{\mathrm{IP}} = 4$

LP-relaxation

$$
\begin{array}{lrcrcrcrcrcrr}
z^* & = & \min & 2x_1 & + & 3x_2 & + & x_3 & + & 4x_4 & & & [\mathbf{c}^T\mathbf{x}] \\
[\text{LP1}] & & \text{s.t.} & 3x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & = & 5 & [\mathbf{Ax} = \mathbf{b}] \\
& & & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 & [\mathbf{x} \in X] \\
& & & & & 0 & \leq & \multicolumn{3}{r}{x_1, x_2, x_3, x_4} & \leq & 1 & [\mathbf{x} \in X]
\end{array}
$$

$$
X = \texttt{conv} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} = \texttt{conv} \, \{\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^6\}
$$

$$
= \left\{ \mathbf{x} \in \Re^4 \,\middle|\, \mathbf{x} = \textstyle\sum_{p=1}^{6} \lambda_p \bar{\mathbf{x}}^p; \;\; \textstyle\sum_{p=1}^{6} \lambda_p = 1; \;\; \lambda_p \geq 0, p = 1, \ldots, 6 \right\}
$$

$$[\text{LP2}] \quad z^* = \min \quad 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 + 7\lambda_5 + 5\lambda_6$$

$$\begin{aligned} \text{s.t.} \quad 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 + 4\lambda_5 + 5\lambda_6 &= 5 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 &= 1 \\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6 &\geq 0 \end{aligned}$$

Start columns: $\lambda_1, \lambda_2, \lambda_3$

[LP2]

$$\begin{aligned} z^* \leq \min \quad & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 \\ \text{s.t.} \quad & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1,\ \lambda_2,\ \lambda_3 \geq 0 \end{aligned}$$

[DLP2]

$$\begin{aligned} z^* \leq \max \quad & 5\pi + q \\ \text{då} \quad & 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \end{aligned}$$

Solution: $\bar{\boldsymbol{\lambda}} = (1, 0, 0)$, $\bar{\pi} = -2, \quad \bar{q} = 15$

Reduced costs

$$
\begin{aligned}
& \min_{\mathbf{x}\in X} (\mathbf{c} - \mathbf{A}^T\bar{\pi})^T\mathbf{x} - \bar{q} \\
= \quad & \min_{p=1,\dots,6} (\mathbf{c} - \mathbf{A}^T\bar{\pi})^T\bar{\mathbf{x}}^p - \bar{q} \\
= \quad & \min_{p=1,\dots,6} \{[(2,3,1,4) - (3,2,3,2)\cdot(-2)]\,\bar{\mathbf{x}}^p - 15\} \\
= \quad & \min\{0,0,1,-1,0,0\} = -1 < 0
\end{aligned}
$$

New extreme point in [LP1]: $\bar{\mathbf{x}}^4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

Column in [LP2]: $\begin{pmatrix} \mathbf{c}^T\bar{\mathbf{x}}^4 \\ \mathbf{A}\bar{\mathbf{x}}^4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

New, extended problem

[LP2]

$$
\begin{array}{rrcl}
z^* \le \min & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 & & \\
\text{s.t.} & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 & = & 5 \\
& \lambda_1 + \ \lambda_2 + \ \lambda_3 + \ \lambda_4 & = & 1 \\
& \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 & \ge & 0
\end{array}
$$

[DLP2]

$$
\begin{array}{rrcl}
z^* \le \max & 5\pi + q & & \\
\text{s.t.} & 5\pi + q & \le & 5 \\
& 6\pi + q & \le & 3 \\
& 5\pi + q & \le & 6 \\
& 5\pi + q & \le & 4
\end{array}
$$

Solution: $\bar{\boldsymbol{\lambda}} = (0, 0, 0, 1),$ $\bar{\pi} = -1, \quad \bar{q} = 9$

Reduced costs: $\min\limits_{p=1,\ldots,6} \{(5, 5, 4, 6)\, \bar{\mathbf{x}}^p - 9\} = \min\{1, 0, 2, 0, 2, 1\} = 0$

Optimal solution to [LP2] and [LP1]

$\boldsymbol{\lambda}^* = (0,0,0,1,0,0), \qquad \pi^* = -1, \quad q^* = 9$

$$\Rightarrow \mathbf{x}^* = \bar{\mathbf{x}}^4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \mathbf{x}^*_{\mathrm{IP}}, \qquad z^* = 4 = z^*_{\mathrm{IP}}$$

It was a coincidence that the solution was integral!

I general, the solution $\mathbf{x}^*$ to [LP1] can contain fractional variable values.

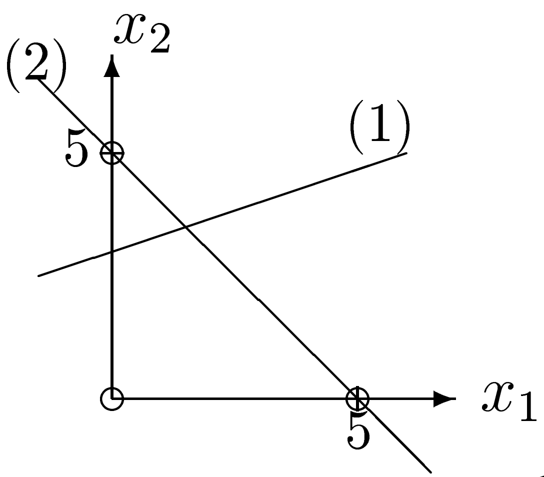
Solution to [IP]

We need to find an integral solution (not certainly an optimal solution to [IP]) among the columns generated, i.e., solve

$$\min \left\{ (2,3,1,4)\mathbf{x} \,\middle|\, (3,2,3,2)\mathbf{x} = 5, \ \mathbf{x} \in \{\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \bar{\mathbf{x}}^3, \bar{\mathbf{x}}^4\} \right\}$$

Numerical example of Dantzig-Wolfe decomposition

$$
\begin{array}{lrcrcrcl}
\min & & x_1 & - & 3x_2 & & & (0) \\
\text{då} & - & x_1 & + & 2x_2 & \leq & 6 & (1) \quad \text{(complicating)} \\
 & & x_1 & + & x_2 & \leq & 5 & (2) \\
 & & x_1 & , & x_2 & \geq & 0 & (3)
\end{array}
$$



$$
\begin{aligned}
X &= \left\{ \mathbf{x} \in \Re^2_+ \,\middle|\, x_1 + x_2 \leq 5 \right\} \\
&= \mathtt{conv} \left\{ (0,0)^T, (0,5)^T, (5,0)^T \right\}
\end{aligned}
$$

Complete DW-master problem

$$\mathbf{x} \in X \Leftrightarrow \left[\begin{array}{l} \mathbf{x} = \lambda_1(0,0)^T + \lambda_2(0,5)^T + \lambda_3(5,0)^T = \begin{pmatrix} 5\lambda_3 \\ 5\lambda_2 \end{pmatrix} \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \right]$$

$$\begin{array}{lrcrcrcr} \min & - & 15\lambda_2 & + & 5\lambda_3 & & & (0) \\ \text{då} & & 10\lambda_2 & - & 5\lambda_3 & \leq & 6 & (1) \\ & & \multicolumn{3}{r}{\lambda_1 + \lambda_2 + \lambda_3} & = & 1 & \\ & & \multicolumn{3}{r}{\lambda_1, \lambda_2, \lambda_3} & \geq & 0 & \end{array}$$

The first master problem is constructed from the points $(0,0)$ and $(0,5)$ (corresponds to λ_1 and λ_2)

Iteration 1

$$\begin{array}{lrcll} \min & -\quad 15\lambda_2 & & & (0) \\ \text{s.t.} & 10\lambda_2 & \leq & 6 & (1) \\ & \lambda_1 + \lambda_2 & = & 1 & \\ & \lambda_1, \lambda_2 & \geq & 0 & \end{array}$$

Solution: $\lambda_1 = \frac{2}{5}, \lambda_2 = \frac{3}{5}$

Dual solution: $\pi = -\frac{3}{2}, q = 0$

Least reduced cost:

$$\min_{\mathbf{x}\in X} \left[(\mathbf{c}^T - \pi\mathbf{A})\mathbf{x} - q\right] = \min_{\mathbf{x}\in X} \left(\left[(1,-3) - (-\tfrac{3}{2})(-1,2)\right]\mathbf{x} - 0\right)$$

$$= \min\{-\tfrac{1}{2}x_1 \mid x_1 + x_2 \leq 5; x_1, x_2 \geq 0\} = -\tfrac{5}{2} < 0 \quad \Rightarrow \quad \bar{\mathbf{x}} = (5,0)^T$$

New column:
$$\begin{array}{r} \mathbf{c}^T\bar{\mathbf{x}} = (1,-3)(5,0)^T = 5 \\ \mathbf{A}\bar{\mathbf{x}} = (-1,2)(5,0)^T = -5 \end{array} \quad \Rightarrow \quad \begin{pmatrix} 5 \\ \hline -5 \\ 1 \end{pmatrix}$$

Iteration 2

$$\begin{array}{rrcl} \min & -15\lambda_2 + 5\lambda_3 & & \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 & \leq & 6 \\ & \lambda_1 + \lambda_2 + \lambda_3 & = & 1 \\ & \lambda_1, \lambda_2, \lambda_3 & \geq & 0 \end{array}$$

Solution:	$\lambda_1 = 0, \lambda_2 = \frac{11}{15}, \lambda_3 = \frac{4}{15}$
Dual solution:	$\pi = -\frac{4}{3}, q = -\frac{5}{3}$

Least reduced cost:

$$\min_{\mathbf{x}\in X} \left[(\mathbf{c}^T - \pi\mathbf{A})\mathbf{x} - q\right] = \min_{\mathbf{x}\in X} \left(\left[(1,-3) - (-\tfrac{4}{3})(-1,2)\right]\mathbf{x} - (-\tfrac{5}{3})\right)$$

$$= \min\{-\tfrac{1}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{5}{3} \mid x_1 + x_2 \leq 5; x_1, x_2 \geq 0\} = 0$$

Optimal solution: $\lambda_1^* = 0, \lambda_2^* = \frac{11}{15}, \lambda_3^* = \frac{4}{15}$

$\Rightarrow \mathbf{x}^* = (5\lambda_3, 5\lambda_2)^T = (\frac{4}{3}, \frac{11}{3})^T$ $\qquad z^* = \frac{4}{3} - 3 \cdot \frac{11}{3} = -\left(9 + \frac{2}{3}\right)$

Block-angular structure

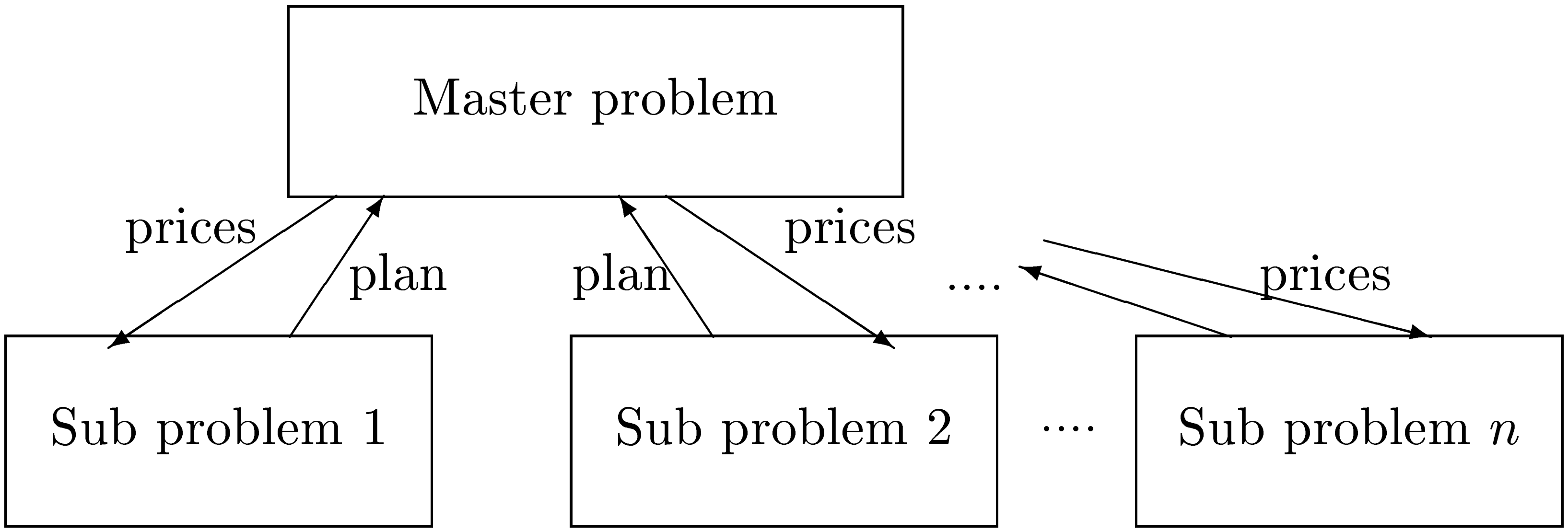
$$
\begin{array}{rcccccccccl}
\max & c_1x_1 & + & c_2x_2 & + & \cdots & + & c_nx_n & & & \\
\text{s.t.} & A_1x_1 & + & A_2x_2 & + & \cdots & + & A_nx_n & \leq & a & \mid \pi \\
 & B_1x_1 & & & & & & & \leq & b_1 & \mid x_1 \in X_1 \\
 & & & B_2x_2 & & & & & \leq & b_2 & \mid x_2 \in X_2 \\
 & & & & & \cdots & & & . & . & \\
 & & & & & & & B_nx_n & \leq & b_n & \mid x_n \in X_n \\
 & & & & & & & x_1, x_2, \cdots, x_n & \geq & 0 &
\end{array}
$$

$$X = X_1 \times X_2 \times \ldots \times X_n$$

Dantzig-Wolfe decomposition

Decentralized planning:

• The main office (master problem) sets prizes ($\boldsymbol{\pi}$) for the common resources (complicating constraints).
• The departments (sub problems) suggest (production) plans ($A_j\bar{x}_j^p$ or $A_j\tilde{x}_j^r$) based on the prices given and their own conditions.
• The main office mixes the suggested plans optimally and sets new prices.



Find feasible solutions (right–hand–side allocation)

Let $\bar{\lambda}_p^j$, $p \in \mathcal{P}$, and $\bar{\mu}_r^j$, $r \in \mathcal{R}$, $j = 1, \ldots, n$, be a feasible and (almost) optimal solution to the master problem. A good feasible x-solution is then given by (for all j):

$$\begin{array}{rrcl} \max & c_j x_j & & \\ \text{s.t.} & A_j x_j & \leq & \displaystyle\sum_{p \in \mathcal{P}} \bar{\lambda}_p^j (A_j \bar{x}_j^p) + \sum_{r \in \mathcal{R}} \bar{\mu}_r^j (A_j \tilde{x}_j^r) \\ & B_j x_j & \leq & b_j \\ & x_j & \geq & 0 \qquad\qquad\qquad [X_j = \{x_j \geq 0 \mid B_j x_j \leq b_j\}] \end{array}$$

since then the following holds

$$\sum_{j=1}^{n} A_j x_j \leq \sum_{j=1}^{n} A_j \underbrace{\left(\sum_{p \in \mathcal{P}} \bar{\lambda}_p^j \bar{x}_j^p + \sum_{r \in \mathcal{R}} \bar{\mu}_r^j \tilde{x}_j^r \right)}_{x_j \in X_j} \leq a$$

Estimates of the optimal objective value

$$z^* = \min\{c^T x \mid Ax = b,\ x \in X\}$$

$$\begin{array}{rrcll}
z^* = \min & \sum_{p\in\mathcal{P}} \lambda_p(\mathbf{c}^T\bar{\mathbf{x}}^p) + \sum_{r\in\mathcal{R}} \mu_r(\mathbf{c}^T\tilde{\mathbf{x}}^r) & & & \\
\text{s.t.} & \sum_{p\in\mathcal{P}} \lambda_p(\mathbf{A}\bar{\mathbf{x}}^p) + \sum_{r\in\mathcal{R}} \mu_r(\mathbf{A}\tilde{\mathbf{x}}^r) & = & \mathbf{b} & \mid \pi \\
 & \sum_{p\in\mathcal{P}} \lambda_p & = & 1 & \mid q \\
 & \lambda_p, \mu_r & \geq & 0,\ p \in \mathcal{P}, r \in \mathcal{R} &
\end{array}$$

$$\begin{array}{rlcll}
z^* \leq \bar{z} = \mathbf{b}^T\bar{\boldsymbol{\pi}} + \bar{q} = \max\limits_{\boldsymbol{\pi},q} & \mathbf{b}^T\boldsymbol{\pi} + q & & & \\
\text{s.t.} & (\mathbf{A}\bar{\mathbf{x}}^p)^T\boldsymbol{\pi} + q & \leq & (\mathbf{c}^T\bar{\mathbf{x}}^p), & p \in \bar{\mathcal{P}} \subseteq \mathcal{P} \\
 & (\mathbf{A}\tilde{\mathbf{x}}^r)^T\boldsymbol{\pi} & \leq & (\mathbf{c}^T\tilde{\mathbf{x}}^r), & r \in \bar{\mathcal{R}} \subseteq \mathcal{R}
\end{array}$$

Let λ_p^*, $p \in \mathcal{P}$, och μ_r^*, $r \in \mathcal{R}$, be an optimal solution to the complete master problem, and let $(\bar{\boldsymbol{\pi}}, \bar{q})$ be an optimal dual solution when the columns in $\bar{\mathcal{P}}$ and $\bar{\mathcal{R}}$ only, are generated. Multiply the right hand side of the primal ($\mathbf{b}$ resp. 1) by $\bar{\boldsymbol{\pi}}$ resp. $\bar{q}$ $\Rightarrow$

$$0 \geq z^* - \bar{z} = z^* - \mathbf{b}^T\bar{\boldsymbol{\pi}} - 1 \cdot \bar{q} =$$
$$\sum_{p\in\mathcal{P}} \lambda_p^* \left[(\mathbf{c}^T\bar{\mathbf{x}}^p) - (\mathbf{A}\bar{\mathbf{x}}^p)^T\bar{\boldsymbol{\pi}} - \bar{q}\right] + \sum_{r\in\mathcal{R}} \mu_r^* \left[(\mathbf{c}^T\tilde{\mathbf{x}}^r) - (\mathbf{A}\tilde{\mathbf{x}}^r)^T\bar{\boldsymbol{\pi}}\right] \geq$$
$$\min_{p\in\mathcal{P}} \left[(\mathbf{c}^T\bar{\mathbf{x}}^p) - (\mathbf{A}\bar{\mathbf{x}}^p)^T\bar{\boldsymbol{\pi}} - \bar{q}\right] + \sum_{r\in\mathcal{R}} \mu_r^* \min_{s\in\mathcal{R}} \left[(\mathbf{c}^T\tilde{\mathbf{x}}^s) - (\mathbf{A}\tilde{\mathbf{x}}^s)^T\bar{\boldsymbol{\pi}}\right]$$

If the subproblem has an unbounded solution no optimistic estimate can be computed in this iteration, otherwise it holds that:

$$\min_{s\in\mathcal{R}} \left[(\mathbf{c}^T\tilde{\mathbf{x}}^s) - (\mathbf{A}\tilde{\mathbf{x}}^s)^T\bar{\boldsymbol{\pi}}\right] \geq 0$$

$\Rightarrow$

$$\bar{z} \geq z^* \geq \bar{z} + \min_{p\in\mathcal{P}} \left[(\mathbf{c} - \mathbf{A}^T\bar{\boldsymbol{\pi}})^T\bar{\mathbf{x}}^p - \bar{q}\right] = \bar{z} + \min_{\mathbf{x}\in X} (\mathbf{c} - \mathbf{A}^T\bar{\boldsymbol{\pi}})^T\mathbf{x} - \bar{q} = \underline{z}$$

Konvergens

Since each polyhedron X has a finite number of extreme points and rays, only a finite number of columns can be generated. When all the columns that will be generated has been generated, the solution to the master problem will also be a solution to the original linear problem. For each new column that is added to the master problem, its optimal objective value will decrease (or be kept constant). Hence, the pessimistic estimate $\bar{z}_t$ will converge monotonely to z^*.

The optimistic estimate $\underline{z}_t$ also converges, but not surely monotonely. At the iteration (t) when an optimal solution to the compete master problem is received, $\underline{z}_t$ will equal $\bar{z}_t$.

Stopping criterion: $\bar{z}_t - \underline{z}_t^* \leq \varepsilon$, where $\underline{z}_t^* = \max_{s=1,\dots,t} \underline{z}_s$ and $\varepsilon > 0$