Mathematics of maintenance planning optimization

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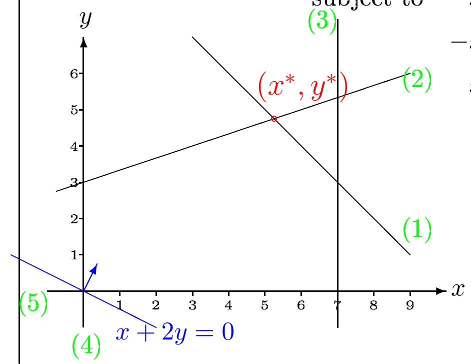
A linear continuous optimization model

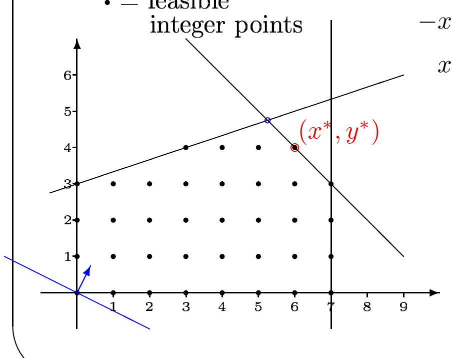
$$
\begin{array}{lrcrcrl}
\text{maximize} & x & + & 2y & & & \\
\text{subject to} & x & + & y & \leq & 10 & (1)\\
 & -x & + & 3y & \leq & 9 & (2)\\
 & x & & & \leq & 7 & (3)\\
 & & & x,y & \geq & 0 & (4,5)
\end{array}
$$



$$(x^*, y^*) = \begin{pmatrix} 5\frac{1}{4} \\ 4\frac{3}{4} \end{pmatrix}$$

A linear integer model

$$
\begin{array}{lrcrcrl}
\text{maximize} & x & + & 2y & & & \\
\text{subject to} & x & + & y & \leq & 10 & (1)\\
 & -x & + & 3y & \leq & 9 & (2)\\
 & x & & & \leq & 7 & (3)\\
 & & & x,y & \geq & 0 & (4,5)\\
 & & & x,y & & \text{integer} &
\end{array}
$$

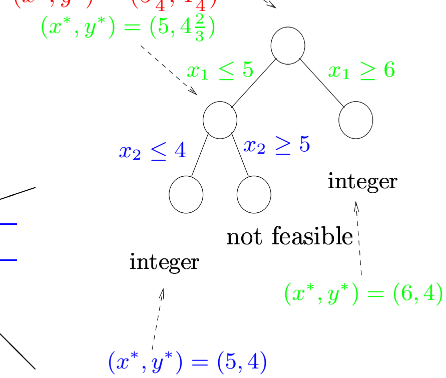


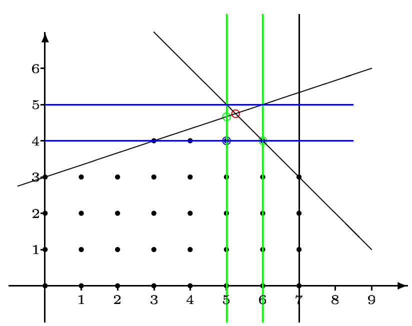
$$(x^*, y^*) = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

Standard algorithm (in e.g. Cplex or Xpress-MP)

Relax integrality requirements ⇒
linear, continuous problem ⇒ $(x^*, y^*) = (5\frac{1}{4}, 4\frac{3}{4})$

Search tree: branch over fractional variable values





In the worst case...

- The one module model has $T \approx 50$ time steps

$\Rightarrow$ 50 integer variables: $z_0, \ldots, z_{49}$

$\Rightarrow$ $2^{50} \approx 10^{15}$ branches

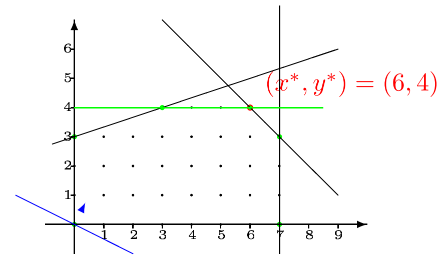
- Solve one continuous problem in 10^{-6} seconds $\Rightarrow$ 10^9seconds $\approx$ 30 years (10^{-9} seconds $\Rightarrow \approx$ 1.5 weeks)

- It is not really this bad for us, but:

  Better to generate facets so that all extreme points become integral

Ultimate goal

- Find all facets $\Rightarrow$ no integrality requirements needed



- *Polyhedron*: $P = \{x \in \Re^n \mid a_i x \leq b_i, i = 1 \ldots, k\}$
- $\pi x \leq \pi_0$ is a *valid inequality* for P if it holds for all $x \in P$
- *Face*: $F = \{x \in P \,|\, \pi x = \pi_0\}$ if $\pi x \leq \pi_0$ is a valid ineq. for P
- *Facet*: face of dimension $n - 1$

The mathematical model for one module

$$\begin{array}{llll} \text{minimize} & \displaystyle\sum_{t=1}^{T-1} \left(\sum_{i \in \mathcal{N}} c_{it} x_{it} + d_t z_t \right), & & \\ \text{subject to} & \displaystyle\sum_{t=\ell}^{T_i+\ell-1} x_{it} \geq 1, & \ell = 0, \ldots, T - T_i, & i \in \mathcal{N}, \\ & 0 \leq x_{it} \leq z_t, & t = 0, \ldots, T-1, & i \in \mathcal{N}, \\ & z_t \in \{0, 1\}, & t = 0, \ldots, T-1. & \end{array}$$

- Use the structure to find *classes of facets*
- Logical consequence of the "natural" constraints plus "smart" roundings

A small example

Two components ($N = 2$) with lives $T_1 = 3$ and $T_2 = 4$ and $T = 5$ time steps.

$$\begin{array}{ll} \text{minimize} & x_{11} + x_{12} + 2x_{13} + x_{14} + x_{21} + 100x_{22} + 100x_{23} + x_{24} \\ & +10z_1 + 10z_2 + z_3 + 10z_4 \\ \text{subject to} & x_{11} + x_{12} + x_{13} \geq 1 \\ & x_{12} + x_{13} + x_{14} \geq 1 \\ & x_{21} + x_{22} + x_{23} + x_{24} \geq 1 \\ & 0 \leq x_{it} \leq z_t, \quad t = 0, \ldots, 4, \quad i \in \{1, 2\} \\ & z_t \in \{0, 1\}, \quad t = 0, \ldots, 4. \end{array}$$

Optimal solution: $x^* = \begin{pmatrix} 0\ 0\ 1\ 0 \\ 1\ 0\ 0\ 0 \end{pmatrix}$ $\quad z^* = (1\ 0\ 1\ 0)$ $\quad$ Value: 14

Continuous relaxation

$$\begin{array}{rl} \text{minimize} & x_{11}+x_{12}+2x_{13}+x_{14}+x_{21}+100x_{22}+100x_{23}+x_{24} \\ & +10z_1+10z_2+z_3+10z_4 \\ \text{subject to} & x_{11}+x_{12}+x_{13} \geq 1 \\ & x_{12}+x_{13}+x_{14} \geq 1 \\ & x_{21}+x_{22}+x_{23}+x_{24} \geq 1 \\ & 0 \leq x_{it} \leq z_t \leq 1, \quad t=0,\dots,4, \quad i \in \{1,2\} \end{array}$$

Optimal solution: $x^* = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$ $\quad z^* = (\frac{1}{2}\ 0\ \frac{1}{2}\ \frac{1}{2})$ $\quad$ Value: $13\frac{1}{2}$

A class of maintenance facets

For the components p and q such that the lives T_p and T_q fulfil

$$2 \leq T_q \leq T_p - 1 \leq 2 \cdot (T_q - 1)$$

a class of facets is defined according to:

$$z_\ell + \sum_{t=\ell+1}^{\ell+T_p-2} (x_{pt} + x_{qt}) + z_{\ell+T_p-1} \geq 2, \qquad \ell = 1, \dots, T - T_p.$$

For the example ($T_1 = 3$, $T_2 = 4$, $T = 5$, $p = 2$, $q = 1$):

$$2 \leq T_1 = 3 \leq T_2 - 1 = 3 \leq 2 \cdot (T_1 - 1) = 4$$

A facet is given by the inequality

$$z_1 + x_{12} + x_{13} + x_{22} + x_{23} + z_4 \geq 2$$

Construction

$$\begin{array}{l} x_{11}+x_{12}+x_{13} \geq 1 \\ x_{12}+x_{13}+x_{14} \geq 1 \\ x_{21}+x_{22}+x_{23}+x_{24} \geq 1 \end{array}$$

Aggregate $\Rightarrow$ $\quad x_{11} + 2x_{12} + 2x_{13} + x_{14} + x_{21} + x_{22} + x_{23} + x_{24} \geq 3 \qquad (1)$

$$\begin{array}{l} x_{11} \leq z_1 \\ x_{14} \leq z_4 \\ x_{21} \leq z_1 \\ x_{24} \leq z_4 \end{array}$$

Aggregate $\Rightarrow$ $\quad 2z_1 + 2z_4 \geq x_{11} + x_{14} + x_{21} + x_{24} \qquad (2)$

(1) and (2) $\Rightarrow$ $\quad 2z_1 + 2x_{12} + 2x_{13} + x_{22} + x_{23} + 2z_4 \geq 3 \qquad (3)$

Construction cont'd

Multiply (3) with $\frac{1}{2}$: $\quad z_1 + x_{12} + x_{13} + \frac{1}{2}x_{22} + \frac{1}{2}x_{23} + z_4 \geq \frac{3}{2} \qquad (4)$

Round-up the coefficients of the LHS to the nearest integer (OK?!):

$$z_1 + x_{12} + x_{13} + x_{22} + x_{23} + z_4 \geq \frac{3}{2}$$

All numbers in the LHS are integral in a feasible solution to the integer program $\Rightarrow$ Round-up the RHS to the nearest integer:

$$z_1 + x_{12} + x_{13} + x_{22} + x_{23} + z_4 \geq 2$$

We have shown that this inequality is *valid* for the *maintenance polytope.*

To show that it defines a *facet* takes a little more ...

The continuous relaxation plus the valid inequality

$$
\begin{array}{ll}
\text{minimize} & x_{11} + x_{12} + 2x_{13} + x_{14} + x_{21} + 100x_{22} + 100x_{23} + x_{24} \\
 & +10z_1 + 10z_2 + z_3 + 10z_4 \\
\text{subject to} & x_{11} + x_{12} + x_{13} \geq 1 \\
 & x_{12} + x_{13} + x_{14} \geq 1 \\
 & x_{21} + x_{22} + x_{23} + x_{24} \geq 1 \\
 & z_1 + x_{12} + x_{13} + x_{22} + x_{23} + z_4 \geq 2 \\
 & 0 \leq x_{it} \leq z_t \leq 1, \quad t = 0, \ldots, 4, \quad i \in \{1, 2\}
\end{array}
$$

Optimal solution: $x^* = \begin{pmatrix} 0\ 0\ 1\ 0 \\ 1\ 0\ 0\ 0 \end{pmatrix}$ $\qquad z^* = (1\ 0\ 1\ 0)$ $\qquad$ Value: 14