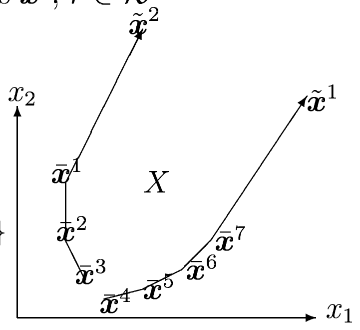
Cutting Plane, Column generation and Dantzig–Wolfe decomposition

26 September 2008

Formulation of LP on column generation form—Dantzig–Wolfe decomposition

Let $X = \{\boldsymbol{x} \in \mathbb{R}^n_+ \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$ (or $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$) be a polyhedron with the *extreme points* $\bar{\boldsymbol{x}}^p$, $p \in \mathcal{P}$ and the *extreme recession directions* $\tilde{\boldsymbol{x}}^r$, $r \in \mathcal{R}$



$\mathcal{P} = \{1, 2, \ldots, 7\}$

$\mathcal{R} = \{1, 2\}$

$$\boldsymbol{x} \in X \iff \begin{pmatrix} \boldsymbol{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\boldsymbol{x}}^p + \sum_{r \in \mathcal{R}} \mu_r \tilde{\boldsymbol{x}}^r \\ \sum_{p \in \mathcal{P}} \lambda_p = 1 \\ \lambda_p \geq 0, \quad p \in \mathcal{P} \\ \mu_r \geq 0, \quad r \in \mathcal{R} \end{pmatrix}$$

$\boldsymbol{x} \in X$ is a convex combination of the extreme points plus a conical combination of the extreme directions

This *inner representation* of the set X can be used to reformulate a linear optimization problem according to the *Dantzig-Wolfe decomposition principle*, which is then solved by column generation.

An LP and its complete master problem

$$\begin{aligned} \text{[LP1]} \quad z^* = \text{minimum } & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to } & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \text{ ("simple" constraints)} \\ & \boldsymbol{D}\boldsymbol{x} = \boldsymbol{d} \text{ (complicating constraints)} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Let $X = \{\, \boldsymbol{x} \geq 0 \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \,\}$ with the extreme points $\bar{\boldsymbol{x}}^p$, $p \in \mathcal{P}$ and the extreme directions $\tilde{\boldsymbol{x}}^r$, $r \in \mathcal{R} \Longrightarrow$

$$
\begin{aligned}
\text{[LP2]}\ z^* = \min\ & \sum_{p\in\mathcal{P}} \lambda_p(\boldsymbol{c}^{\mathrm{T}}\bar{\boldsymbol{x}}^p) + \sum_{r\in\mathcal{R}} \mu_r(\boldsymbol{c}^{\mathrm{T}}\tilde{\boldsymbol{x}}^r) \\
\text{s.t.}\ & \sum_{p\in\mathcal{P}} \lambda_p(\boldsymbol{D}\bar{\boldsymbol{x}}^p) + \sum_{r\in\mathcal{R}} \mu_r(\boldsymbol{D}\tilde{\boldsymbol{x}}^r) = \boldsymbol{d} \quad | \, \boldsymbol{\pi} \\
& \sum_{p\in\mathcal{P}} \lambda_p = 1 \quad | \, q \\
& \lambda_p, \mu_r \geq 0,\ \forall p, r
\end{aligned}
$$

Number of constraints in [LP2] equals to "the number of constraints in $\boldsymbol{D}\boldsymbol{x} = \boldsymbol{d}$" + 1

Number of columns very large (# extreme pts./dirs. to X)

The dual of [LP2] is given by (not all extreme pts./dirs. found yet: $\bar{\mathcal{P}} \subseteq \mathcal{P}$; $\bar{\mathcal{R}} \subseteq \mathcal{R}$)

$$
\begin{aligned}
\text{[DLP2]}\ z^* \leq \max_{(\boldsymbol{\pi},q)}\ & \boldsymbol{d}^{\mathrm{T}}\boldsymbol{\pi} + q \\
\text{s.t.}\ & (\boldsymbol{D}\bar{\boldsymbol{x}}^p)^{\mathrm{T}}\boldsymbol{\pi} + q \leq (\boldsymbol{c}^{\mathrm{T}}\bar{\boldsymbol{x}}^p), \quad p \in \bar{\mathcal{P}} \quad | \, \lambda_p \\
& (\boldsymbol{D}\tilde{\boldsymbol{x}}^r)^{\mathrm{T}}\boldsymbol{\pi} \leq (\boldsymbol{c}^{\mathrm{T}}\tilde{\boldsymbol{x}}^r), \quad r \in \bar{\mathcal{R}} \quad | \, \mu_r
\end{aligned}
$$

with solutions $(\bar{\boldsymbol{\pi}}, \bar{q})$

Reduced cost for the variable λ_p, $p \in \mathcal{P} \setminus \bar{\mathcal{P}}$ is given by

$$(\boldsymbol{c}^{\mathrm{T}}\bar{\boldsymbol{x}}^p) - (\boldsymbol{D}\bar{\boldsymbol{x}}^p)^{\mathrm{T}}\bar{\boldsymbol{\pi}} - \bar{q} = (\boldsymbol{c} - \boldsymbol{D}^{\mathrm{T}}\bar{\boldsymbol{\pi}})^{\mathrm{T}}\bar{\boldsymbol{x}}^p - \bar{q}$$

Reduced cost for the variable μ_r, $r \in \mathcal{R} \setminus \bar{\mathcal{R}}$ is given by

$$(\boldsymbol{c}^{\mathrm{T}}\tilde{\boldsymbol{x}}^r) - (\boldsymbol{D}\tilde{\boldsymbol{x}}^r)^{\mathrm{T}}\bar{\boldsymbol{\pi}} = (\boldsymbol{c} - \boldsymbol{D}^{\mathrm{T}}\bar{\boldsymbol{\pi}})^{\mathrm{T}}\tilde{\boldsymbol{x}}^r$$

Column generation

The least reduced cost is found by solving the subproblem

$$\min_{\boldsymbol{x}\in X} (\boldsymbol{c} - \boldsymbol{D}^{\mathrm{T}}\boldsymbol{\pi})^{\mathrm{T}}\boldsymbol{x} \quad \left(\text{alt:} \quad \min_{\boldsymbol{x}\in X} (\boldsymbol{c} - \boldsymbol{D}^{\mathrm{T}}\bar{\boldsymbol{\pi}})^{\mathrm{T}}\boldsymbol{x} - \bar{q}\right)$$

Gives as solution an extreme point, $\bar{\boldsymbol{x}}^p$, or an extreme direction $\tilde{\boldsymbol{x}}^r$

$\Longrightarrow$ a new column in [LP2]: (if < 0)

Either $\begin{pmatrix} \boldsymbol{c}^{\mathrm{T}}\bar{\boldsymbol{x}}^p \\ \boldsymbol{D}\bar{\boldsymbol{x}}^p \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \boldsymbol{c}^{\mathrm{T}}\tilde{\boldsymbol{x}}^r \\ \boldsymbol{D}\tilde{\boldsymbol{x}}^r \\ 0 \end{pmatrix}$ enters the problem and improves the solution

Example

$$
\begin{aligned}
z^*_{\mathrm{IP}} &= \min\ 2x_1 + 3x_2 + x_3 + 4x_4 \\
\text{[IP]} \quad \text{s.t.}\ & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 \qquad | \, \boldsymbol{D}\boldsymbol{x} = \boldsymbol{d} \\
& x_1 + x_2 + x_3 + x_4 = 2 \\
& x_1 \quad x_2 \quad x_3 \quad x_4 \in \{0, 1\}
\end{aligned}
$$

$$X = \left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\} = \{\bar{\boldsymbol{x}}^1, \ldots, \bar{\boldsymbol{x}}^6\}$$

Optimal solution: $\boldsymbol{x}^*_{\mathrm{IP}} = (0, 1, 1, 0)^{\mathrm{T}}$ $\qquad z^*_{\mathrm{IP}} = 4$

LP-relaxation

$$
\begin{array}{rrl}
z^* & = \min & 2x_1 + 3x_2 + x_3 + 4x_4 \qquad [\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}] \\
\text{[LP1]} & \text{s.t.} & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5\,[\boldsymbol{D}\boldsymbol{x} = \boldsymbol{d}] \\
 & & x_1 + x_2 + x_3 + x_4 = 2 \quad [\boldsymbol{x} \in X] \\
 & 0 \le & x_1 \quad\; x_2 \quad\; x_3 \quad\; x_4 \le 1 \quad [\boldsymbol{x} \in X]
\end{array}
$$

$$
X = \operatorname{conv}\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\} = \operatorname{conv}\{\bar{\boldsymbol{x}}^1, \dots, \bar{\boldsymbol{x}}^6\}
$$

$$
= \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \boldsymbol{x} = \textstyle\sum_{p=1}^{6} \lambda_p \bar{\boldsymbol{x}}^p;\ \sum_{p=1}^{6} \lambda_p = 1;\ \lambda_p \ge 0,\ p = 1, \dots, 6 \right\}
$$

$$
\begin{array}{rrl}
\text{[LP2]} \quad z^* = & \min & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 + 7\lambda_5 + 5\lambda_6 \\
 & \text{s.t.} & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 + 4\lambda_5 + 5\lambda_6 = 5 \\
 & & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 \\
 & & \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6 \ge 0
\end{array}
$$

Start columns: $\lambda_1, \lambda_2, \lambda_3$

[LP2]

$$
\begin{array}{rl}
z^* \le \min & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 \\
\text{s.t.} & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 = 5 \\
 & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\
 & \lambda_1,\ \lambda_2,\ \lambda_3 \ge 0
\end{array}
$$

[DLP2]

$$
\begin{array}{rl}
z^* \le \max & 5\pi + q \\
\text{s.t.} & 5\pi + q \le 5 \\
 & 6\pi + q \le 3 \\
 & 5\pi + q \le 6
\end{array}
$$

Solution: $\bar{\boldsymbol{\lambda}} = (1, 0, 0)^{\mathrm{T}}$, $\quad \bar{\pi} = -2, \quad \bar{q} = 15$

Reduced costs

$$
\begin{aligned}
& \min_{\boldsymbol{x} \in X} (\boldsymbol{c} - \boldsymbol{D}^{\mathrm{T}}\bar{\pi})^{\mathrm{T}}\boldsymbol{x} - \bar{q} \\
= & \min_{\boldsymbol{x} \in X} \{[(2,3,1,4) - (3,2,3,2)\cdot(-2)]\,\boldsymbol{x} - 15\} \\
= & \min_{\sum x = 2} \{(8,7,7,8)\boldsymbol{x} - 15\} = -1 < 0
\end{aligned}
$$

New extreme point in [LP1]: $\bar{\boldsymbol{x}}^4 = (0, 1, 1, 0)^{\mathrm{T}}$

Column in [LP2]: $\begin{pmatrix} \boldsymbol{c}^{\mathrm{T}}\bar{\boldsymbol{x}}^4 \\ \boldsymbol{A}\bar{\boldsymbol{x}}^4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

New, extended problem

[LP2]

$$
\begin{array}{rl}
z^* \le \min & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 \\
\text{s.t.} & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 = 5 \\
 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\
 & \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 \ge 0
\end{array}
$$

[DLP2]

$$
\begin{array}{rl}
z^* \le \max & 5\pi + q \\
\text{s.t.} & 5\pi + q \le 5 \\
 & 6\pi + q \le 3 \\
 & 5\pi + q \le 6 \\
 & 5\pi + q \le 4
\end{array}
$$

Solution:

$\bar{\boldsymbol{\lambda}} = (0, 0, 0, 1)^{\mathrm{T}}$, $\quad \bar{\pi} = -1, \quad \bar{q} = 9$

Reduced costs: $\min_{\boldsymbol{x} \in X} \{(5, 5, 4, 6)\,\boldsymbol{x} - 9\} = 0$

Optimal solution to [LP2] and [LP1]

$\boldsymbol{\lambda}^* = (0,0,0,1,0,0)^{\mathrm{T}}, \qquad \pi^* = -1, \quad q^* = 9$

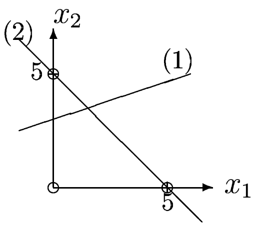
$\Longrightarrow \boldsymbol{x}^* = \bar{\boldsymbol{x}}^4 = (0,1,1,0)^{\mathrm{T}} = \boldsymbol{x}^*_{\mathrm{IP}}, \qquad z^* = 4 = z^*_{\mathrm{IP}}$

It was a coincidence that the solution was integral!

In general, the solution $\boldsymbol{x}^*$ to [LP1] can have fractional variable values. In this case we could have found an integral (not necessary optimal) solution among the extrempoints generated so far.

Numerical example of Dantzig-Wolfe decomposition

$$
\begin{array}{llllll}
\min & x_1 - 3x_2 & & & (0) & \\
\text{s.t.} & -x_1 + 2x_2 & \leq & 6 & (1) & \text{(complicating)} \\
 & x_1 + x_2 & \leq & 5 & (2) & \\
 & x_1 \;,\; x_2 & \geq & 0 & (3) &
\end{array}
$$



$$
\begin{aligned}
X &= \left\{ \boldsymbol{x} \in \mathbb{R}^2_+ \,\middle|\, x_1 + x_2 \leq 5 \right\} \\
&= \operatorname{conv}\left\{ (0,0)^{\mathrm{T}}, (0,5)^{\mathrm{T}}, (5,0)^{\mathrm{T}} \right\}
\end{aligned}
$$

Complete DW-master problem

$$
\boldsymbol{x} \in X \iff \begin{cases} \boldsymbol{x} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\lambda_3 \\ 5\lambda_2 \end{pmatrix} \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{cases}
$$

$$
\begin{array}{llr}
\min & -15\lambda_2 + 5\lambda_3 & (0) \\
\text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 & (1) \\
 & \lambda_1 + \lambda_2 + \lambda_3 = 1 & \\
 & \lambda_1, \lambda_2, \lambda_3 \geq 0 &
\end{array}
$$

The first master problem is constructed from the points $(0,0)^{\mathrm{T}}$ and $(0,5)^{\mathrm{T}}$ (corresponds to λ_1 and λ_2)

Iteration 1

$$
\begin{array}{llr}
\min & -15\lambda_2 & (0) \\
\text{s.t.} & 10\lambda_2 \leq 6 & (1) \\
 & \lambda_1 + \lambda_2 = 1 & \\
 & \lambda_1, \lambda_2 \geq 0 &
\end{array}
$$

Solution: $\boldsymbol{\lambda} = (\frac{2}{5}, \frac{3}{5})^{\mathrm{T}}$

Dual solution: $\pi = -\frac{3}{2}, q = 0$

Least reduced cost: $\min_{\boldsymbol{x} \in X} \left[(\boldsymbol{c}^{\mathrm{T}} - \pi \boldsymbol{D})\boldsymbol{x} - q \right]$

$= \min_{\boldsymbol{x} \in X} \left(\left[(1,-3) - (-\frac{3}{2})(-1,2) \right] \boldsymbol{x} - 0 \right)$

$= \min \left\{ -\frac{1}{2}x_1 \mid x_1 + x_2 \leq 5; \boldsymbol{x} \geq \boldsymbol{0}^2 \right\} = -\frac{5}{2} < 0 \Longrightarrow \bar{\boldsymbol{x}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

New column: $\boldsymbol{c}^{\mathrm{T}}\bar{\boldsymbol{x}} = (1,-3)(5,0)^{\mathrm{T}} = 5$

$\boldsymbol{D}\bar{\boldsymbol{x}} = (-1,2)(5,0)^{\mathrm{T}} = -5 \qquad \Longrightarrow \begin{pmatrix} 5 \\ \hline -5 \\ 1 \end{pmatrix}$

Block-angular structure

$$
\begin{array}{llllll}
\max & c_1^{\mathrm{T}}\boldsymbol{x}_1 + & c_2^{\mathrm{T}}\boldsymbol{x}_2 + \cdots + & c_n^{\mathrm{T}}\boldsymbol{x}_n & & \\
\text{s.t.} & \boldsymbol{D}_1\boldsymbol{x}_1 + & \boldsymbol{D}_2\boldsymbol{x}_2 + \cdots + & \boldsymbol{D}_n\boldsymbol{x}_n & \leq \boldsymbol{d} & \mid \boldsymbol{\pi} \\
& \boldsymbol{A}_1\boldsymbol{x}_1 & & & \leq \boldsymbol{b}_1 & \mid \boldsymbol{x}_1 \in X_1 \\
& & \boldsymbol{A}_2\boldsymbol{x}_2 & & \leq \boldsymbol{b}_2 & \mid \boldsymbol{x}_2 \in X_2 \\
& & \cdots & & \cdot\ \cdot & \\
& & & \boldsymbol{A}_n\boldsymbol{x}_n & \leq \boldsymbol{b}_n & \mid \boldsymbol{x}_n \in X_n \\
& & \boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n & & \geq \boldsymbol{0} &
\end{array}
$$

$$X = X_1 \times X_2 \times \ldots \times X_n$$

Iteration 2

$$
\begin{array}{ll}
\min & -15\lambda_2 + 5\lambda_3 \\
\text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 \\
& \lambda_1 + \lambda_2 + \lambda_3 = 1 \\
& \lambda_1, \lambda_2, \lambda_3 \geq 0
\end{array}
$$

Solution: $\boldsymbol{\lambda} = (0, \frac{11}{15}, \frac{4}{15})^{\mathrm{T}}$

Dual solution: $\pi = -\frac{4}{3}, q = -\frac{5}{3}$

Least reduced cost: $\min_{\boldsymbol{x} \in X} \left[(\boldsymbol{c}^{\mathrm{T}} - \pi \boldsymbol{D})\boldsymbol{x} - q\right]$

$= \min_{\boldsymbol{x} \in X} \left(\left[(1, -3) - (-\frac{4}{3})(-1, 2)\right] \boldsymbol{x} - (-\frac{5}{3})\right)$

$= \min \{-\frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{5}{3} \mid x_1 + x_2 \leq 5; \boldsymbol{x} \geq \boldsymbol{0}^2\} = 0$

Optimal solution: $\boldsymbol{\lambda}^* = (0, \frac{11}{15}, \frac{4}{15})^{\mathrm{T}}$

$\Longrightarrow \boldsymbol{x}^* = (5\lambda_3, 5\lambda_2)^{\mathrm{T}} = (\frac{4}{3}, \frac{11}{3})^{\mathrm{T}}; \quad z^* = \frac{4}{3} - 3 \cdot \frac{11}{3} = -9\frac{2}{3}$

DW decomposition as decentralized planning

- Main office (master problem) sets prizes ($\boldsymbol{\pi}$) for the common resources (complicating constraints).
- Departments (subproblems) suggest (production) plans ($\boldsymbol{D}_j \bar{\boldsymbol{x}}_j^p$) based on given prices.
- Main office mixes suggested plans optimally; new prices.