Lecture 12: Benders decomposition and Branch–and–price

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Benders decomposition for mixed-integer optimization problems—Lasdon (1970)

- Model:

$$\begin{aligned} \text{minimum} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} + f(\mathbf{y}), \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{F}(\mathbf{y}) \geq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n, \quad \mathbf{y} \in S. \end{aligned}$$

- The variables $\mathbf{y}$ are "difficult" because:
  - the set S may be complicated, like $S \subseteq \{0, 1\}^p$
  - f and/or $\mathbf{F}$ may be nonlinear
  - the vector $\mathbf{F}(\mathbf{y})$ may cover every row, while the problem in $\mathbf{x}$ for fixed $\mathbf{y}$ may separate
- The problem is *linear*, possibly separable in $\mathbf{x}$; "easy"

Example: Block-angular structure in x, binary constraints on y, linear in x, nonlinear in y

$$
\begin{aligned}
\min\ & \mathbf{c}_1^{\mathrm{T}}\mathbf{x}_1 + \cdots + \mathbf{c}_n^{\mathrm{T}}\mathbf{x}_n + f(\mathbf{y}) \\
\text{s.t.}\ & \mathbf{A}_1\mathbf{x}_1 \qquad\qquad\qquad + \mathbf{F}_1(\mathbf{y}) \geq \mathbf{b}_1 \\
& \qquad\qquad \ddots \qquad\qquad\qquad \vdots \qquad \vdots \\
& \qquad\qquad\qquad \mathbf{A}_n\mathbf{x}_n + \mathbf{F}_n(\mathbf{y}) \geq \mathbf{b}_n \\
& \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \qquad\qquad \geq \mathbf{0} \\
& \qquad\qquad\qquad\qquad\quad \mathbf{y} \in \{0,1\}^p
\end{aligned}
$$

- **Typical application:** Multi-stage stochastic programming (optimization under uncertainty)
  - Some parameters (constants) are uncertain
  - Choose $\mathbf{y}$ (e.g., investment) such that an *expected* cost over time is minimized
  - Uncertainty in data is represented by future *scenarios* (ℓ)
  - Variables $\mathbf{x}_\ell$ represent future activities
  - $\mathbf{y}$ must be chosen before the outcome of the uncertain parameters is known
  - Choose $\mathbf{y}$ s.t. the expected value over scenarios ℓ of the future optimization over $\mathbf{x}_\ell$ ($\Rightarrow \mathbf{x}_\ell(\mathbf{y})$) is the best

A two-stage stochastic program

$$
\begin{array}{lrcrcll}
\min & \displaystyle\sum_{\ell\in\mathcal{L}} p^{\ell}\cdot \mathbf{c}_{\ell}^{\mathrm{T}}\mathbf{x}_{\ell} & + & \mathbf{d}^{\mathrm{T}}\mathbf{y} & & & \\
\text{s.t.} & \mathbf{A}_{\ell}\mathbf{x}_{\ell} & + & \mathbf{T}_{\ell}\mathbf{y} & = & \mathbf{b}_{\ell}, & \ell\in\mathcal{L} \\
 & \mathbf{x}_{\ell} & & & \geq & \mathbf{0}, & \ell\in\mathcal{L} \\
 & & & \mathbf{y} & \in & Y &
\end{array}
$$

- **Solution idea:** Temporarily fix $\mathbf{y}$, solve the remaining problem over $\mathbf{x}$ parameterized over $\mathbf{y} \Rightarrow$ solution $\mathbf{x}(\mathbf{y})$. Utilize the problem structure to improve the guess of an optimal value of $\mathbf{y}$. Repeat.

- Similar to minimizing a function η over two vectors, $\mathbf{v}$ and $\mathbf{w}$:

$$\inf_{\mathbf{v},\mathbf{w}} \eta(\mathbf{v},\mathbf{w}) = \inf_{\mathbf{v}} \xi(\mathbf{v}), \text{ where } \xi(\mathbf{v}) = \inf_{\mathbf{w}} \eta(\mathbf{v},\mathbf{w}), \mathbf{v} \in \mathbb{R}^m.$$

- In effect, we substitute the variable $\mathbf{w}$ by always minimizing over it, and work with the remaining problem in $\mathbf{v}$

- **Benders decomposition:** construct an approximation of this problem over $\mathbf{v}$ by utilizing LP duality

- If the problem over $\mathbf{y}$ is also linear

$\Rightarrow$ cutting plane methods from above

- Benders decomposition is more general: Solves problems with positive duality gaps!

- Benders decomposition does *not* rely on the existence of optimal Lagrange multipliers and strong duality

The Benders sub- and master problems

- The model revisited:

$$\begin{aligned} \text{minimum} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} + f(\mathbf{y}), \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{F}(\mathbf{y}) \geq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n, \quad \mathbf{y} \in S. \end{aligned}$$

- Which values of $\mathbf{y}$ are feasible?
  Choose $\mathbf{y} \in S$ such that the remaining problem in $\mathbf{x}$ is feasible

- Choose $\mathbf{y}$ in the set

$$R := \{\, \mathbf{y} \in S \mid \exists \mathbf{x} \geq \mathbf{0}^n \text{ with } \mathbf{A}\mathbf{x} \geq \mathbf{b} - \mathbf{F}(\mathbf{y}) \,\}$$

Apply Farkas' Lemma to this system, or rather to the equivalent system (with $\mathbf{y}$ fixed):

$$\mathbf{A}\mathbf{x} - \mathbf{s} = \mathbf{b} - \mathbf{F}(\mathbf{y})$$
$$\mathbf{x} \geq \mathbf{0}^n, \;\; \mathbf{s} \geq \mathbf{0}^m$$

- From Farkas' Lemma, $\mathbf{y} \in R$ if and only if

$$\mathbf{A}^{\mathrm{T}}\mathbf{u} \leq \mathbf{0}^n, \; \mathbf{u} \geq \mathbf{0}^m \quad \Longrightarrow \quad [\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}}\mathbf{u} \leq 0$$

This means that $\mathbf{y} \in R$ if and only if $[\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}}\mathbf{u}_i^r \leq 0$ holds for every extreme direction $\mathbf{u}_i^r$, $i = 1, \ldots, n_r$ of the polyhedral cone $C = \{ \mathbf{u} \in \mathbb{R}_+^m \mid \mathbf{A}^{\mathrm{T}}\mathbf{u} \leq \mathbf{0}^n \}$

- We here made good use of the Representation Theorem for a polyhedral cone

- Given $\mathbf{y} \in R$, the optimal value in *Benders' subproblem* is

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimum}} \;\; & \mathbf{c}^{\mathrm{T}}\mathbf{x}, \\ \text{subject to} \;\; & \mathbf{A}\mathbf{x} \geq \mathbf{b} - \mathbf{F}(\mathbf{y}), \\ & \mathbf{x} \geq \mathbf{0}^{n}, \end{aligned}$$

- By LP duality, this equals

$$\begin{aligned} \underset{\mathbf{u}}{\text{maximum}} \;\; & [\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}}\mathbf{u}, \\ \text{subject to} \;\; & \mathbf{A}^{\mathrm{T}}\mathbf{u} \leq \mathbf{c}, \\ & \mathbf{u} \geq \mathbf{0}^{m}, \end{aligned}$$

provided that the first problem has a finite solution

- We prefer the dual formulation, since its constraints do not depend on $\mathbf{y}$

- Moreover, the *extreme directions* of its feasible set are given by the vectors $\mathbf{u}_i^r$, $i = 1, \dots, n_r$, discussed above

- Let $\mathbf{u}_i^p$, $i = 1, \dots, n_p$, denote the *extreme points* of this set

- This completes the subproblem

- Let's now study the *restricted master* problem (RMP) of Benders' algorithm

- The original model:

$$\begin{aligned} \text{minimum} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} + f(\mathbf{y}), \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{F}(\mathbf{y}) \geq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n, \quad \mathbf{y} \in S. \end{aligned}$$

- This is equivalent to

$$\begin{aligned} & \min_{\mathbf{y}\in S} \left\{ f(\mathbf{y}) + \min_{\mathbf{x}} \{ \mathbf{c}^{\mathrm{T}}\mathbf{x} \mid \mathbf{A}\mathbf{x} \geq \mathbf{b} - \mathbf{F}(\mathbf{y}); \mathbf{x} \geq \mathbf{0}^n \} \right\} \\ = & \min_{\mathbf{y}\in R} \left\{ f(\mathbf{y}) + \max_{\mathbf{u}} \{ [\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}}\mathbf{u} \mid \mathbf{A}^{\mathrm{T}}\mathbf{u} \leq \mathbf{c};\ \mathbf{u} \geq \mathbf{0}^m \} \right\} \\ = & \min_{\mathbf{y}\in R} \left\{ f(\mathbf{y}) + \max_{i=1,\ldots n_p} \{ [\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}}\mathbf{u}_i^p \} \right\} \end{aligned}$$

... continued ...

$$\min_{\mathbf{y}\in R} \left\{ f(\mathbf{y}) + \max_{i=1,\ldots n_p} \{ [\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}} \mathbf{u}_i^p \} \right\}$$

$$\begin{aligned}
= \min\ & z \\
\text{s.t. } & z \geq f(\mathbf{y}) + [\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}} \mathbf{u}_i^p, \quad i = 1, \ldots, n_p, \\
& \mathbf{y} \in R, \\
= \min\ & z \\
\text{s.t. } & z \geq f(\mathbf{y}) + [\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}} \mathbf{u}_i^p, \quad i = 1, \ldots, n_p, \\
& 0 \geq [\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}} \mathbf{u}_i^r, \quad i = 1, \ldots, n_r, \\
& \mathbf{y} \in S.
\end{aligned}$$

- Suppose that not the whole sets of constraints in the latter problem is known
- This means that not all extreme points and directions for the dual problem are known
- Replace "$i = 1, \ldots, n_p$" with "$i \in I_1$" and "$i = 1, \ldots, n_r$" with "$i \in I_2$" where $I_1 \subset \{1, \ldots, n_p\}$ and $I_2 \subset \{1, \ldots, n_r\}$
- Since not all constraints are included, we get a lower bound on the optimal value of the original problem

- Suppose that $(z^0, \mathbf{y}^0)$ is a finite optimal solution to this problem
- To check if this is indeed an optimal solution to the original problem: check for the most violated constraint, which we
  - either satisfy, $\Rightarrow$ $\mathbf{y}^0$ is optimal
  - or not, $\Rightarrow$ include this new constraint, extending either the set I_1 or I_2, and possibly improving the lower bound.

- The search for a new constraint is solving the dual of Benders' subproblem with $\mathbf{y} = \mathbf{y}^0$:

$$
\begin{aligned}
\underset{\mathbf{u}}{\text{maximum}} \quad & [\mathbf{b} - \mathbf{F}(\mathbf{y}^0)]^{\mathrm{T}}\mathbf{u}, \\
\text{subject to} \quad & \mathbf{A}^{\mathrm{T}}\mathbf{u} \leq \mathbf{c}, \\
& \mathbf{u} \geq \mathbf{0}^m,
\end{aligned}
$$

  $\Rightarrow$ a new extreme point or direction due to a new objective

- The solution $\mathbf{u}(\mathbf{y}^0)$ to this (dual) problem corresponds to a *feasible* (primal) solution $(\mathbf{x}(\mathbf{y}^0), \mathbf{y}^0)$ to the original problem, and therefore also an *upper bound* on the optimal value, provided that it is finite

- If this problem has an unbounded solution, then it is unbounded along an extreme direction: $[\mathbf{b} - \mathbf{F}(\mathbf{y}^0)]^{\mathrm{T}}\mathbf{u}_i^r > 0$

$\Rightarrow$ Add the constr. $0 \geq [\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}}\mathbf{u}_i^r$ to RMP (enlarge I_2)

- Suppose instead that the optimal solution is finite:

$\Rightarrow$ Let $\mathbf{u}_i^p$ be an optimal extreme point
If $z^0 < f(\mathbf{y}^0) + [\mathbf{b} - \mathbf{F}(\mathbf{y}^0)]^{\mathrm{T}}\mathbf{u}_i^p$, add the constraint $z \geq f(\mathbf{y}) + [\mathbf{b} - \mathbf{F}(\mathbf{y})]^{\mathrm{T}}\mathbf{u}_i^p$ to RMP (enlarge I_1)

- If $z^0 \geq f(\mathbf{y}^0) + [\mathbf{b} - \mathbf{F}(\mathbf{y}^0)]^{\mathrm{T}}\mathbf{u}_i^p$ then equality must hold ($>$ cannot happen—why?)

$\Rightarrow$ We then have an optimal solution to the original problem and terminate.

Convergence

- Suppose that S is closed and bounded and that f and $\mathbf{F}$ are both continuous on S. Then, provided that the computations are exact, we terminate in a finite number of iterations with an optimal solution.
- Proof is due to the finite number of constraints in the complete master problem, that is, the number of extreme points and directions in any polyhedron.
- A numerical example of the use of Benders decomposition is found in Lasdon (1970, Sections 7.3.3–7.3.5).

- Note the resemblance to the Dantzig–Wolfe algorithm! In fact, if f and $\mathbf{F}$ both are linear, then they coincide, in the sense that (the duals of) their subproblems and restricted master problems are identical!
- Modern implementations of the Dantzig–Wolfe and Benders algorithms are inexact, that is, at least their RMP:s are not solved exactly.
- Moreover, their RMP:s are often restricted such that there is an additional "box constraint" added. This constraint forces the solution to the next RMP to be relatively close to the previous one.

- The effect is that of a stabilization; otherwise, there is a risk that the sequence of solutions to the RMP:s “jump about,” and convergence becomes slow as the optimal solution is approached.
- This was observed quite early on with the Dantzig–Wolfe algorithm, which even can be enriched with non-linear “penalty” terms in the RMP to further stabilize convergence.
- In any case, convergence holds also under these modifications, except perhaps for the finiteness.

Branch and Price

—

Branch and Bound with column generation

A linear integer problem

$$
\begin{aligned}
z^* = \min \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & 2x_1 + 2x_2 \geq 1 \\
& x_1, x_2 \in \{0, 1\},
\end{aligned}
$$

$$x^* = (1, 0), \;\; z^* = 1$$

$$
\begin{aligned}
z^*_{LP} = \min \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & 2x_1 + 2x_2 \geq 1 \\
& x_1, x_2 \in [0, 1]
\end{aligned}
$$

$$x^*_{LP} = \left(\frac{1}{2}, 0\right), \;\; z^*_{LP} = \frac{1}{2}$$

$$z^*_{LP} \leq z^*$$

About Branch–and–Bound

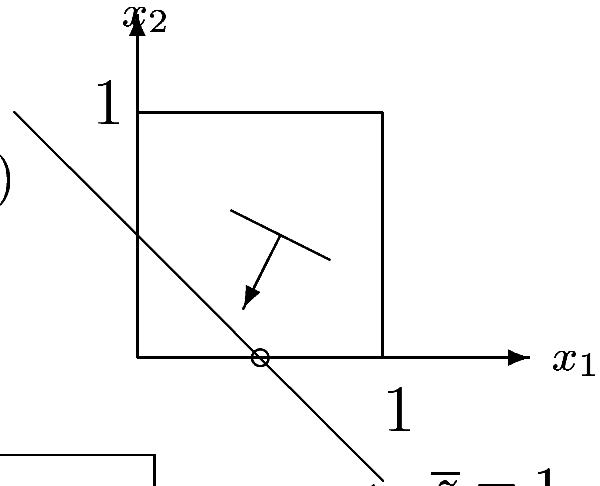
[LP]

$$\begin{aligned} \min\ & x_1 + 2x_2 \\ \text{s.t.}\ & 2x_1 + 2x_2 \geq 1 \\ & x_1, x_2 \in [0,1] \end{aligned}$$

$\underline{z} = 1$

$\Uparrow$

$z_{LP} = \frac{1}{2}$ $\mathbf{x}_{LP} = (\frac{1}{2}, 0)$



$\overline{z} = 1$

LP

$x_1{=}0$ $x_1{=}1$

$\underline{z} = 1$ $z_{LP0} = 1$

$\mathbf{x}_{LP0} = (0, \frac{1}{2})$ LP0

LP1 $z_{LP1} = 1$

$\mathbf{x}_{LP1} = (1, 0)$

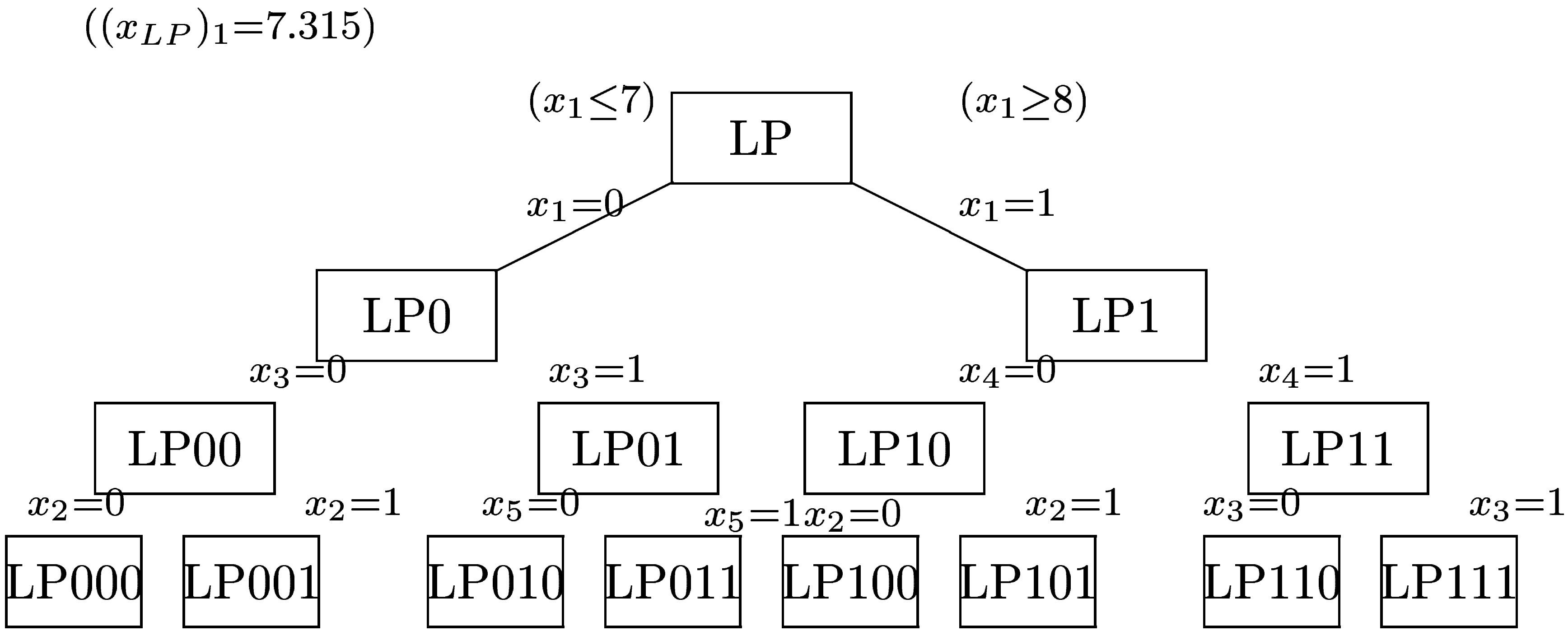
[LP0]

$$\begin{aligned} \min\ & 2x_2 \\ \text{s.t.}\ & 2x_2 \geq 1 \\ & x_2 \in [0,1] \end{aligned}$$

[LP1]

$$\begin{aligned} \min\ & 1 + 2x_2 \\ \text{s.t.}\ & 2 + 2x_2 \geq 1 \\ & x_2 \in [0,1] \end{aligned}$$

A Branch–and–Bound tree



etc.

Branch–and–price for linear 0/1 problems

Consider the DW–column generation setting:

$$
\begin{aligned}
\text{[IP]} \quad z^*_{\mathrm{IP}} = \min \ & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\
\text{s.t. } & \mathbf{Dx} = \mathbf{d} \\
& \mathbf{x} \in X = \{\mathbf{x} \in \mathbb{B}^n \mid \mathbf{Ax} = \mathbf{b}\} = \{\bar{\mathbf{x}}^p \mid p \in \mathcal{P}\}
\end{aligned}
$$

Inner representation (and convexification):

$$
\operatorname{conv} X = \left\{ \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p \,\middle|\, \sum_{p \in \mathcal{P}} \lambda_p = 1; \ \lambda_p \geq 0, \ p \in \mathcal{P} \right\}
$$

Let $c_p = \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p$ and $\mathbf{d}_p = \mathbf{D}\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$.

Stronger formulation—Master problem

$$[\text{CP}] \qquad z^*_{\text{IP}} = z^*_{\text{CP}} = \min \sum_{p \in \mathcal{P}} c_p \lambda_p$$
$$\text{s.t.} \sum_{p \in \mathcal{P}} \mathbf{d}_p \lambda_p = \mathbf{d}$$
$$\sum_{p \in \mathcal{P}} \lambda_p = 1$$
$$\lambda_p \in \{0, 1\}, \qquad p \in \mathcal{P}$$

A continuous relaxation ([CP^{cont}], to $\lambda_p \geq 0$) of [CP] gives the same lower bound as the Lagrangian dual for the constraints $\mathbf{Dx} = \mathbf{d}$. ($z^*_{LP} \leq z^{cont}_{CP} \leq z^*_{CP}$)

The continuous relaxation [LP] of [IP] is never better than any Lagrange dual bound.

Restricted master problem

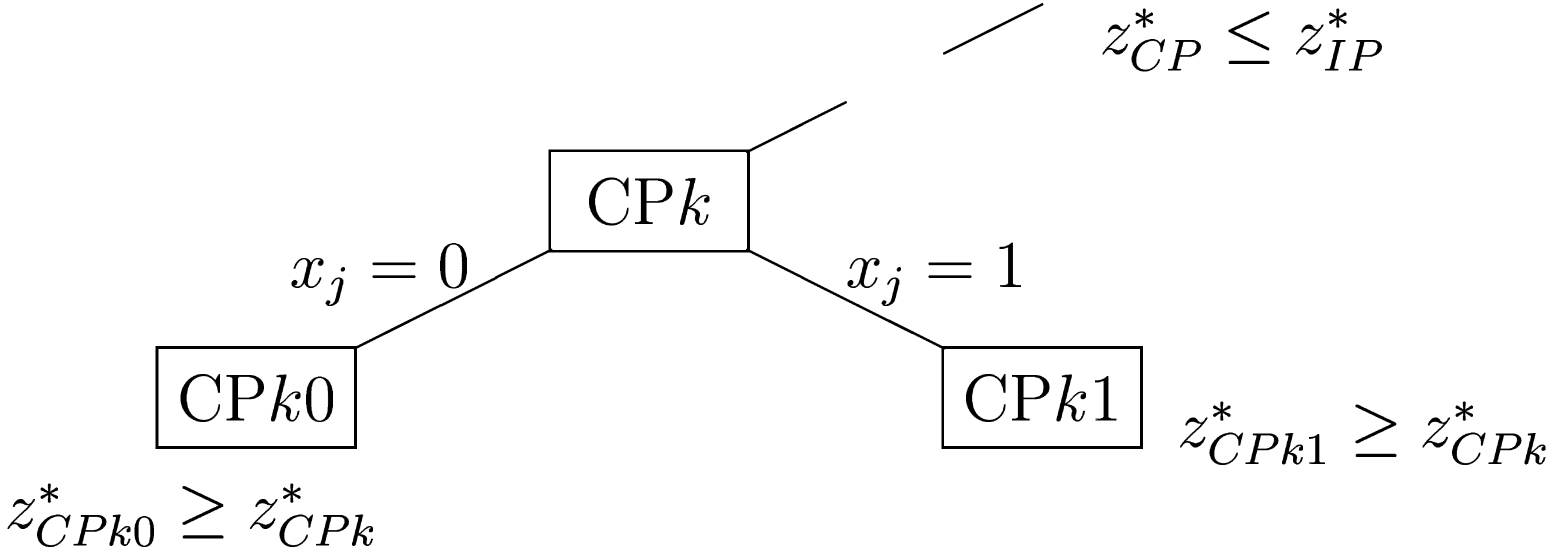
Let $\bar{\mathcal{P}} \subseteq \mathcal{P}$—only a subset of the columns are generated

$$
\begin{aligned}
\overline{[\mathrm{CP}]} \qquad z^*_{\mathrm{CP}} \geq z^{cont}_{\mathrm{CP}} \leq \bar{z}_{\mathrm{CP}} = \min \ & \sum_{p \in \bar{\mathcal{P}}} c_p \lambda_p \\
\text{s.t.} \ & \sum_{p \in \bar{\mathcal{P}}} \mathbf{d}_p \lambda_p = \mathbf{d} \\
& \sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 \qquad (*) \\
& \lambda_p \geq 0, \qquad p \in \bar{\mathcal{P}}
\end{aligned}
$$

- Generate columns $\begin{pmatrix} c_p \\ \mathbf{d}_p \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ until an (almost) optimal solution to $[\mathrm{CP}^{cont}]$, $\widehat{\lambda}_p$ $(p \in \bar{\mathcal{P}})$, is found $\Rightarrow \widehat{\mathbf{x}} = \sum_{p \in \bar{\mathcal{P}}} \widehat{\lambda}_p \bar{\mathbf{x}}^p$

Branching over variable x_j with $0 < \hat{x}_j < 1$

$$
\begin{array}{ccccc}
 & x_j = 0 & \text{or} & x_j = 1 & \\
 & \Updownarrow & & \Updownarrow & \\
 & x_j = \sum\limits_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 0 & & x_j = \sum\limits_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 1 & \\
 & \Downarrow & & \Downarrow & \\
\text{delete col's} & \sum\limits_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 1} \lambda_p = 0 & & \sum\limits_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 1} \lambda_p = 1 & \text{replaces } (*) \\
 & \Updownarrow & & \Updownarrow & \\
\text{replaces } (*) & \sum\limits_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 0} \lambda_p = 1 & & \sum\limits_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 0} \lambda_p = 0 & \text{delete col's}
\end{array}
$$



- In each node (CP, CP0, CP1, ...): Generate columns until (almost) optimal (all reduced costs ≥ 0) or verified infeasible

- If $\mathbf{x}^*_{CPk\ell\ldots}$ feasible $\Longrightarrow z^*_{CPk\ell\ldots} \geq z^*_{IP} \Longrightarrow$ Cut off the branch $(k, \ell, \ldots)$

$\Longrightarrow$ Cut branches $(r, s, \ldots)$ with $z^*_{CPrs\ldots} \geq z^*_{CPk\ell\ldots}$

The column generation subproblem, reduced costs

- $\displaystyle\min_{\mathbf{x}\in X^k} (\mathbf{c}-\mathbf{D}^{\mathrm{T}}\widehat{\boldsymbol{\pi}}^k)^{\mathrm{T}}\mathbf{x}-\widehat{q}^k =: (\mathbf{c}-\mathbf{D}^{\mathrm{T}}\widehat{\boldsymbol{\pi}}^k)^{\mathrm{T}}\bar{\mathbf{x}}^p-\widehat{q}^k =: \bar{c}(\bar{\mathbf{x}}^p)$
- $(\widehat{\boldsymbol{\pi}}^k, \widehat{q}^k)$ is a dual solution to the RMP and $X^k = X \cap \{\mathbf{x} \mid x_j = k\}$, $k \in \{0,1\}$ (etc. down the tree)
- If $\bar{c}(\bar{\mathbf{x}}^p) < 0$ then $\begin{pmatrix} \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p \\ \mathbf{D}\bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ is a new column in [CPk]
- Minimization? $\bar{\mathbf{x}}^r$ is good enough if $\bar{c}(\bar{\mathbf{x}}^r) < 0$
- If $\bar{c}(\bar{\mathbf{x}}^p) \geq 0$ then no more columns are needed to solve [CPk] to optimality.
- Same columns may be generated in different nodes $\Longrightarrow$ create "column pool" to check w.r.t. reduced costs $\bar{c}$

An instance solved by Branch–and–price

$$
\begin{aligned}
z^*_{IP} = \min \quad & x_1 + 2x_2 & = z^*_{CP} \quad \geq z^{cont}_{CP} = z^*_{LP} = \min \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & 2x_1 + 2x_2 \geq 1 & \text{s.t.} \quad & 2x_1 + 2x_2 \geq 1 \\
& x_1, x_2 \in \{0,1\} & & 0 \leq x_1, x_2 \leq 1
\end{aligned}
$$

$$
\text{conv}X = \text{conv}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \lambda_3 + \lambda_4 \\ \lambda_2 + \lambda_4 \end{pmatrix} \middle| \sum_{p=1}^{4} \lambda_p = 1; \begin{array}{r} \lambda_p \geq 0 \\ \forall p \end{array} \right\}
$$

$$
\begin{aligned}
\text{[CP]} \qquad\qquad z^{cont}_{CP} = \min \quad & 2\lambda_2 + \lambda_3 + 3\lambda_4 \\
\text{s.t.} \quad & 2\lambda_2 + 2\lambda_3 + 4\lambda_4 \geq 1 \\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0
\end{aligned}
$$

Start columns: λ_1 and λ_3

Choose e.g., $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, that is, the variables λ_1 and λ_3

$$
\begin{array}{rrcccrr}
z_{CP}^{cont} \leq \min & \lambda_3 & & = & \max & \pi + q \\
\text{s.t.} & 2\lambda_3 \geq 1 & & & \text{s.t.} & q \leq 0 \\
& \lambda_1 + \lambda_3 = 1 & & & & 2\pi + q \leq 1 \\
& \lambda_1, \lambda_3 \geq 0 & & & & \pi \geq 0
\end{array}
$$

Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_3) = (\frac{1}{2}, \frac{1}{2}) \Longrightarrow \widehat{\mathbf{x}} = (\frac{1}{2}, 0)^{\mathrm{T}}$, $\widehat{\pi} = \frac{1}{2}$, $\widehat{q} = 0$
Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(0,1)\mathbf{x}\} = 0 \Longrightarrow$ Optimum for CP!

$$
\begin{array}{lccc}
\text{Fixations:} & x_1 = 0 & \text{or} & x_1 = 1 \\
& \Downarrow & & \Downarrow \\
& \lambda_3 = 0 & & \lambda_1 = 0
\end{array}
$$

Branching, left (CP0): $\lambda_3 = 0$

$$
\begin{aligned}
\min \quad & 0 \\
\text{s.t.} \quad & 0 \geq 1 \\
& \lambda_1 = 1 \\
& \lambda_1 \geq 0
\end{aligned}
\implies
\begin{bmatrix}
\text{infeasible} \\
\Downarrow \\
\text{add} \\
\text{column}
\end{bmatrix}
\implies
\begin{aligned}
z_{CP0} \leq \min \quad & 2\lambda_2 \\
\text{s.t.} \quad & 2\lambda_2 \geq 1 \\
& \lambda_1 + \lambda_2 = 1 \\
& \lambda_1, \lambda_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
= \quad \max \quad & \pi + q \\
\text{s.t.} \quad & q \leq 0 \\
& 2\pi + q \leq 2 \\
& \pi \geq 0
\end{aligned}
$$

Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_2) = (\frac{1}{2}, \frac{1}{2})$

$\implies \widehat{\mathbf{x}} = (0, \frac{1}{2})^{\mathrm{T}}$

$\widehat{\pi} = 1, \quad \widehat{q} = 0$

Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(-1, 0)\mathbf{x} - 0\} = -1 < 0$

$\implies$ New column! (λ_3 or λ_4, but $\lambda_3 \equiv 0$) $\implies$ Choose λ_4

$$z_{CP0} \le \min \; 2\lambda_2 + 3\lambda_4 \quad \text{s.t.} \quad 2\lambda_2 + 4\lambda_4 \ge 1, \quad \lambda_1 + \lambda_2 + \lambda_4 = 1, \quad \lambda_1, \lambda_2, \lambda_4 \ge 0$$

$$= \max \; \pi + q \quad \text{s.t.} \quad q \le 0, \quad 2\pi + q \le 2, \quad 4\pi + q \le 3, \quad \pi \ge 0$$

- Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_3, \widehat{\lambda}_4) = (\frac{3}{4}, 0, \frac{1}{4}) \Longrightarrow \widehat{\mathbf{x}} = (\frac{1}{4}, \frac{1}{4})^{\mathrm{T}}$, $\widehat{\pi} = \frac{3}{4}$, $\widehat{q} = 0$
- Reduced costs: $\min_{\mathbf{x}\in[0,1]^2} \{(-\frac{1}{2}, \frac{1}{2})\mathbf{x}\} = -\frac{1}{2} \Longrightarrow$
- Generate new column: λ_3, but $\lambda_3 \equiv 0 \Longrightarrow$ Optimum for CP0

Branching, right (CP1): $\lambda_1 = 0$

$$z_{CP1} \leq \min \quad \lambda_3 \quad \text{s.t.} \quad 2\lambda_3 \geq 1, \quad \lambda_3 = 1, \quad \lambda_3 \geq 0 \quad = \quad \max \quad \pi + q \quad \text{s.t.} \quad 2\pi + q \leq 1, \quad \pi \geq 0$$

- Solution: $\widehat{\lambda}_3 = 1 \Longrightarrow \widehat{\mathbf{x}} = (1,0)^{\mathrm{T}}$, $\widehat{\pi} = 0$, $\widehat{q} = 1$
- Reduced costs: $\min_{\mathbf{x}\in[0,1]^2} \{(1,2)\mathbf{x} - 1\} = -1 < 0 \Longrightarrow$
- Generate new column: λ_1, but $\lambda_1 \equiv 0 \Longrightarrow$ Optimum for CP1 !!

Branching, left, left: (CP00) $\lambda_2 = \lambda_4 = 0$

CP00: $\lambda_2 = \lambda_3 = \lambda_4 = 0 \Longrightarrow$ infeasible

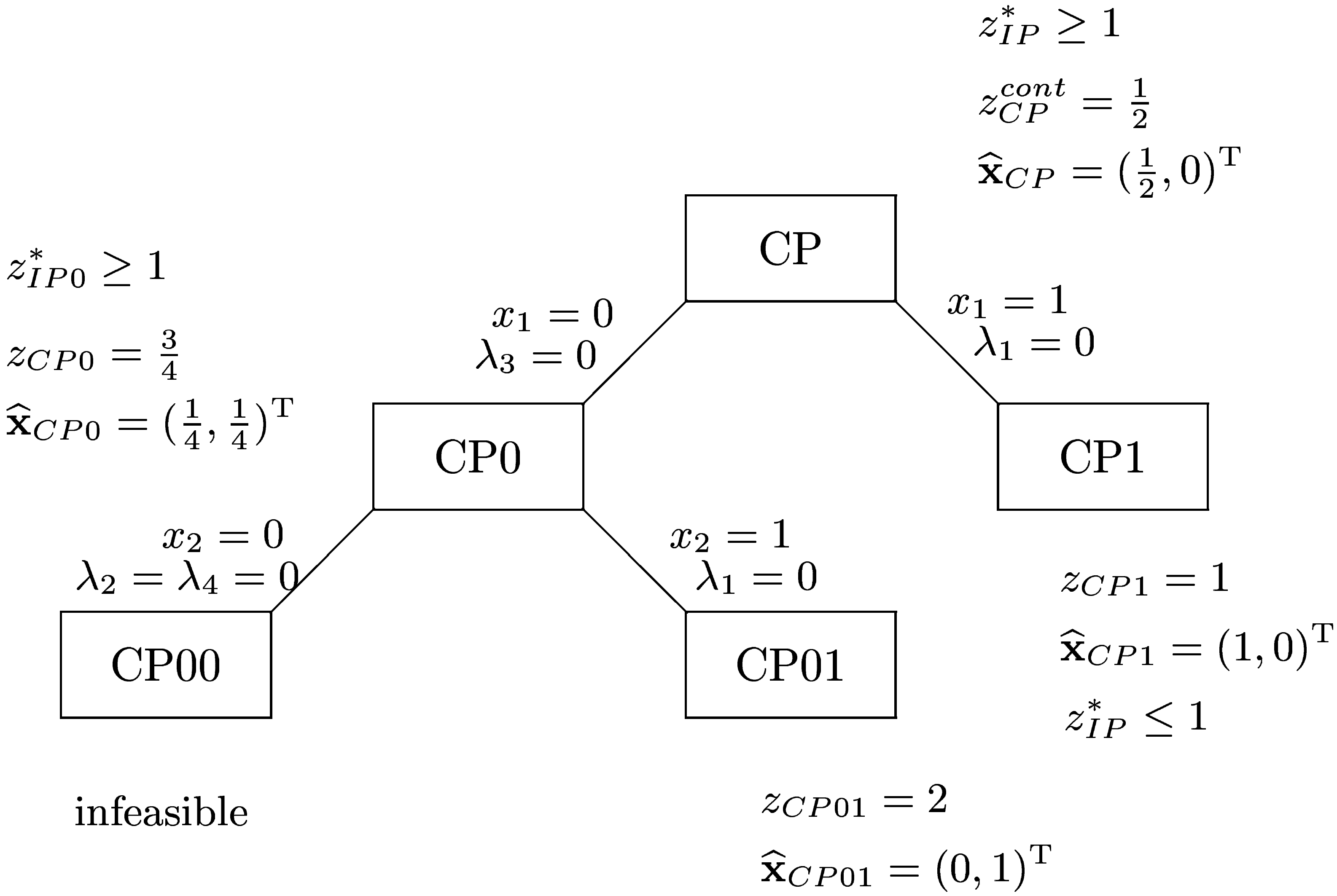
Branching, left, right: (CP01) $\lambda_1 = 0$

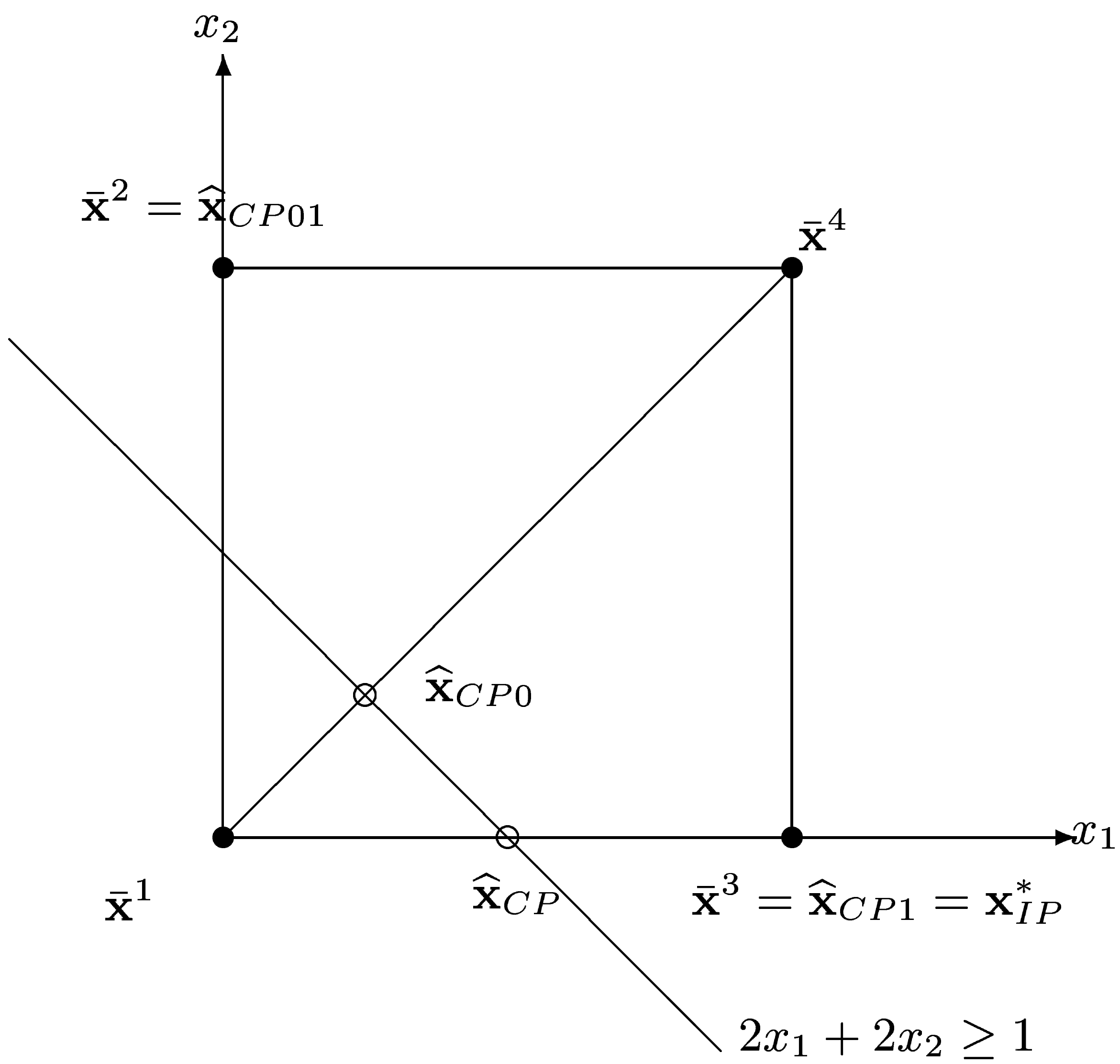
CP01: $\lambda_1 = \lambda_3 = 0$

$$
\begin{aligned}
z_{CP01} \le \min \quad & 2\lambda_2 + 3\lambda_4 & = \quad \max \quad & \pi + q \\
\text{s.t.} \quad & 2\lambda_2 + 4\lambda_4 \ge 1 & \text{s.t.} \quad & 2\pi + q \le 2 \\
& \lambda_2 + \lambda_4 = 1 & & 4\pi + q \le 3 \\
& \lambda_2, \lambda_4 \ge 0 & & \pi \ge 0
\end{aligned}
$$

- Solution: $(\widehat{\lambda}_2, \widehat{\lambda}_4) = (1, 0)^{\mathrm{T}} \Longrightarrow \widehat{\mathbf{x}} = (0, 1)^{\mathrm{T}}$, $\widehat{\pi} = 0$, $\widehat{q} = 2$
- Reduced costs: $\min_{\mathbf{x}\in[0,1]^2} \{(1,2)\mathbf{x} - 2\} = -2 < 0$
  $\Longrightarrow$ Generate new column: λ_1, but $\lambda_1 \equiv 0$
  $\Longrightarrow$ Generate new column: λ_3, but $\lambda_3 \equiv 0$
  $\Longrightarrow$ Optimum for CP01 !!

Branch–and–price tree




x_2
$\bar{\mathbf{x}}^2 = \widehat{\mathbf{x}}_{CP01}$
$\bar{\mathbf{x}}^4$
$\widehat{\mathbf{x}}_{CP0}$
x_1
$\bar{\mathbf{x}}^1$
$\widehat{\mathbf{x}}_{CP}$
$\bar{\mathbf{x}}^3 = \widehat{\mathbf{x}}_{CP1} = \mathbf{x}^*_{IP}$
$2x_1 + 2x_2 \geq 1$