

# Lecture 10: Dantzig–Wolfe decomposition, column generation, and branch–and–price

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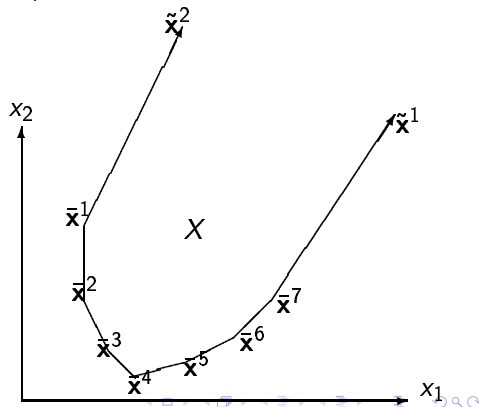
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# Formulation of LP on column generation form: Dantzig–Wolfe decomposition

- ▶ Let  $X = \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{Ax} = \mathbf{b}\}$  (or  $\mathbf{Ax} \leq \mathbf{b}$ ) be a polyhedron with
- ▶ extreme points  $\bar{\mathbf{x}}^p$ ,  $p \in \mathcal{P}$  and
- ▶ extreme recession directions  $\tilde{\mathbf{x}}^r$ ,  $r \in \mathcal{R}$

$$\mathcal{P} = \{1, 2, \dots, 7\}$$

$$\mathcal{R} = \{1, 2\}$$



# Inner representation of the set $X$

$$\mathbf{x} \in X \iff \left( \begin{array}{l} \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p + \sum_{r \in \mathcal{R}} \mu_r \tilde{\mathbf{x}}^r \\ \sum_{p \in \mathcal{P}} \lambda_p = 1 \\ \lambda_p \geq 0, \quad p \in \mathcal{P} \\ \mu_r \geq 0, \quad r \in \mathcal{R} \end{array} \right)$$

- ▶  $\mathbf{x} \in X$  is a *convex* combination of the extreme points plus a *conical* combination of the extreme directions
- ▶ Use this *inner representation* of the set  $X$  to reformulate an LP according to the *Dantzig-Wolfe decomposition principle*
- ▶ Solve by *column generation*

# An LP and its complete master problem

$$\begin{aligned} \text{(LP1)} \quad z^* = \text{minimum } & \mathbf{c}^T \mathbf{x} \\ \text{subject to } & \mathbf{Ax} = \mathbf{b} \quad (\text{"simple" constraints}) \\ & \mathbf{Dx} = \mathbf{d} \quad (\text{complicating constraints}) \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- ▶ Let  $X = \{ \mathbf{x} \geq \mathbf{0} \mid \mathbf{Ax} = \mathbf{b} \}$
- ▶ Extreme points  $\bar{\mathbf{x}}^p, p \in \mathcal{P}$
- ▶ Extreme directions  $\tilde{\mathbf{x}}^r, r \in \mathcal{R}$



# The complete master problem

$$\begin{aligned} \text{(LP2) } z^* = \min \quad & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{c}^T \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{c}^T \tilde{\mathbf{x}}^r) \\ \text{s.t.} \quad & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{D} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} && | \pi \\ & \sum_{p \in \mathcal{P}} \lambda_p = 1 && | q \\ & \lambda_p \geq 0, \quad p \in \mathcal{P} \\ & \mu_r \geq 0, \quad r \in \mathcal{R} \end{aligned}$$

- ▶ Number of constraints in (LP2) equals “the number of constraints in  $\mathbf{D}\mathbf{x} = \mathbf{d}$ ” + 1
- ▶ Number of columns very large ( $\#$  extreme points/directions of  $X$ )

# The LP dual of the (restricted) master problem

- ▶ Assume that not all extreme points/directions are found:  
 $\bar{\mathcal{P}} \subseteq \mathcal{P}; \bar{\mathcal{R}} \subseteq \mathcal{R}$
- ▶ The dual of (LP2-R) is given by

$$\begin{aligned} \text{(DLP2-R)} \quad z^* &\leq \max_{(\boldsymbol{\pi}, q)} \mathbf{d}^T \boldsymbol{\pi} + q \\ \text{s.t.} \quad &(\mathbf{D}\bar{\mathbf{x}}^p)^T \boldsymbol{\pi} + q \leq (\mathbf{c}^T \bar{\mathbf{x}}^p), \quad p \in \bar{\mathcal{P}} \quad | \lambda_p \\ &(\mathbf{D}\tilde{\mathbf{x}}^r)^T \boldsymbol{\pi} \leq (\mathbf{c}^T \tilde{\mathbf{x}}^r), \quad r \in \bar{\mathcal{R}} \quad | \mu_r \end{aligned}$$

with solution  $(\bar{\boldsymbol{\pi}}, \bar{q})$

- ▶ Reduced cost for the variable  $\lambda_p, p \in \mathcal{P} \setminus \bar{\mathcal{P}}$ :  
 $(\mathbf{c}^T \bar{\mathbf{x}}^p) - (\mathbf{D}\bar{\mathbf{x}}^p)^T \bar{\boldsymbol{\pi}} - \bar{q} = (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \bar{\mathbf{x}}^p - \bar{q}$
- ▶ Reduced cost for the variable  $\mu_r, r \in \mathcal{R} \setminus \bar{\mathcal{R}}$ :  
 $(\mathbf{c}^T \tilde{\mathbf{x}}^r) - (\mathbf{D}\tilde{\mathbf{x}}^r)^T \bar{\boldsymbol{\pi}} = (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \tilde{\mathbf{x}}^r$

- ▶ The smallest reduced cost is found by solving the subproblem

$$\min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} \quad \left( \text{alt: } \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} - \bar{q} \right)$$

- ▶ Gives as solution an extreme point,  $\bar{\mathbf{x}}^p$ , or an extreme direction  $\tilde{\mathbf{x}}^r$

⇒ a new column in (LP2) (if the reduced cost  $< 0$ ):

- ▶ Either  $\begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$  or  $\begin{pmatrix} \mathbf{c}^T \tilde{\mathbf{x}}^r \\ \mathbf{D} \tilde{\mathbf{x}}^r \\ 0 \end{pmatrix}$  enters the problem and improves the solution

(IP)

$$\begin{aligned} z_{\text{IP}}^* &= \min 2x_1 + 3x_2 + x_3 + 4x_4 \\ \text{s.t. } & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 \quad | \quad \mathbf{D}\mathbf{x} = \mathbf{d} \\ & x_1 + x_2 + x_3 + x_4 = 2 \\ & x_1 \quad x_2 \quad x_3 \quad x_4 \in \{0, 1\} \end{aligned}$$

$$\blacktriangleright X = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} = \{\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^6\}$$

- ▶ Optimal solution:  $\mathbf{x}_{\text{IP}}^* = (0, 1, 1, 0)^T$
- ▶ Optimal value:  $z_{\text{IP}}^* = 4$



(LP1)

$$\begin{array}{ll}
 z^* = \min & 2x_1 + 3x_2 + x_3 + 4x_4 & [\mathbf{c}^T \mathbf{x}] \\
 \text{s.t.} & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 & [\mathbf{D}\mathbf{x} = \mathbf{d}] \\
 & x_1 + x_2 + x_3 + x_4 = 2 & [\mathbf{x} \in X] \\
 & 0 \leq x_1, x_2, x_3, x_4 \leq 1 & [\mathbf{x} \in X]
 \end{array}$$

$$\blacktriangleright X = \text{conv} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$= \text{conv} \{ \bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^6 \}$$

$$\stackrel{=}{=} \left\{ \mathbf{x} \in \mathbb{R}^4 \mid \mathbf{x} = \sum_{p=1}^6 \lambda_p \bar{\mathbf{x}}^p; \sum_{p=1}^6 \lambda_p = 1; \lambda_p \geq 0, p = 1, \dots, 6 \right\}$$

# The complete master problem and the initial columns

(LP2)

$$\begin{aligned} z^* = \min & \quad 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 + 7\lambda_5 + 5\lambda_6 \\ \text{s.t.} & \quad 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 + 4\lambda_5 + 5\lambda_6 = 5 \\ & \quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 \\ & \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 0 \end{aligned}$$

- ▶ Initial columns:  $\lambda_1, \lambda_2, \lambda_3$

(LP2-R)

$$\begin{aligned} z^* \leq \min & \quad 5\lambda_1 + 3\lambda_2 + 6\lambda_3 \\ \text{s.t.} & \quad 5\lambda_1 + 6\lambda_2 + 5\lambda_3 = 5 \\ & \quad \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \quad \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}$$

(DLP2-R)

$$\begin{aligned} z^* \leq \max & \quad 5\pi + q \\ \text{s.t.} & \quad 5\pi + q \leq 5 \\ & \quad 6\pi + q \leq 3 \\ & \quad 5\pi + q \leq 6 \end{aligned}$$

- ▶ Solution:  $\bar{\lambda} = (1, 0, 0)^T$ ,

$$\bar{\pi} = -2, \quad \bar{q} = 15$$



$$\begin{aligned} \min_{\mathbf{x} \in X} \left\{ (\mathbf{c} - \mathbf{D}^T \bar{\pi})^T \mathbf{x} - \bar{q} \right\} &= \min_{p=1, \dots, 6} \left\{ (\mathbf{c} - \mathbf{D}^T \bar{\pi})^T \bar{\mathbf{x}}^p - \bar{q} \right\} \\ &= \min_{p=1, \dots, 6} \left\{ [(2, 3, 1, 4) - (3, 2, 3, 2) \cdot (-2)] \bar{\mathbf{x}}^p - 15 \right\} \\ &= \min \{0, 0, 1, -1, 0, 0\} = -1 < 0 \end{aligned}$$

- ▶ New extreme point in (LP1):  $\bar{\mathbf{x}}^4 = (0, 1, 1, 0)^T$

- ▶ New column in (LP2-R):  $\begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^4 \\ \mathbf{A} \bar{\mathbf{x}}^4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

(LP2-R)

$$\begin{aligned} z^* \leq \min & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 \\ \text{s.t.} & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

(DLP2-R)

$$\begin{aligned} z^* \leq \max & 5\pi + q \\ \text{s.t.} & 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \\ & 5\pi + q \leq 4 \end{aligned}$$

► Solution:  $\bar{\lambda} = (0, 0, 0, 1)^T$ ,  $\bar{\pi} = -1$ ,  $\bar{q} = 9$

► Reduced costs:

$$\min_{p=1,\dots,6} \{ (5, 5, 4, 6) \bar{x}^p - 9 \} = \min \{ 1, 0, 2, 0, 2, 1 \} = 0$$

# Optimal solution to (LP2) and (LP1)

$$\begin{aligned} &\blacktriangleright \lambda^* = (0, 0, 0, 1, 0, 0)^T, \quad \pi^* = -1, \quad q^* = 9 \\ \implies &\mathbf{x}^* = \bar{\mathbf{x}}^4 = (0, 1, 1, 0)^T = \mathbf{x}_{\text{IP}}^*, \quad z^* = 4 = z_{\text{IP}}^* \end{aligned}$$

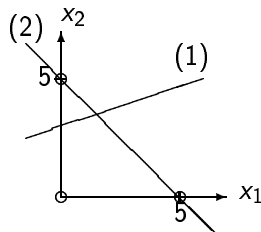
- ▶ A coincidence that the solution was integral!
- ▶ In general, the solution  $\mathbf{x}^*$  to (LP1) may have fractional variable values
- ▶ **Solution to (IP)?**
- ▶ Need to find an integral solution (not certainly an optimal solution to (IP)) among the columns generated, i.e., solve

$$\min \left\{ (2, 3, 1, 4)\mathbf{x} \mid (3, 2, 3, 2)\mathbf{x} = 5, \mathbf{x} \in \{\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \bar{\mathbf{x}}^3, \bar{\mathbf{x}}^4\} \right\}$$

# Numerical example of Dantzig-Wolfe decomposition

$$\begin{array}{llllll} \min & x_1 & - & 3x_2 & & (0) \\ \text{st} & -x_1 & + & 2x_2 & \leq & 6 & (1) & \text{(complicating)} \\ & x_1 & + & x_2 & \leq & 5 & (2) \\ & x_1 & , & x_2 & \geq & 0 & (3) \end{array}$$

$$\begin{aligned} X &= \{ \mathbf{x} \in \mathbb{R}_+^2 \mid x_1 + x_2 \leq 5 \} \\ &= \text{conv} \{ (0, 0)^T, (0, 5)^T, (5, 0)^T \} \end{aligned}$$



# Complete DW master problem



$$\mathbf{x} \in X \iff \left\{ \begin{array}{l} \mathbf{x} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\lambda_3 \\ 5\lambda_2 \end{pmatrix} \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \right\}$$



$$\begin{array}{ll} \min & -15\lambda_2 + 5\lambda_3 & (0) \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 & (1) \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array}$$

- ▶ The first restricted master problem is then constructed from the points  $(0, 0)^T$  and  $(0, 5)^T$  (corresponds to  $\lambda_1$  and  $\lambda_2$ )

# Iteration 1



$$\begin{array}{ll} \min & -15\lambda_2 & (0) \\ \text{s.t.} & 10\lambda_2 \leq 6 & (1) \\ & \lambda_1 + \lambda_2 = 1 \\ & \lambda_1, \lambda_2 \geq 0 \end{array} \quad \left| \begin{array}{l} \text{Solution:} \\ \text{Dual solution:} \end{array} \right. \quad \begin{array}{l} \boldsymbol{\lambda} = \left(\frac{2}{5}, \frac{3}{5}\right)^T \\ \pi = -\frac{3}{2}, q = 0 \end{array}$$

- ▶ Least reduced cost:

$$\min_{\mathbf{x} \in X} [(\mathbf{c}^T - \pi \mathbf{D})\mathbf{x} - q] = \min_{\mathbf{x} \in X} ([ (1, -3) - (-\frac{3}{2})(-1, 2) ] \mathbf{x} - 0)$$

$$= \min \left\{ -\frac{1}{2}x_1 \mid x_1 + x_2 \leq 5; \mathbf{x} \geq \mathbf{0} \right\} = -\frac{5}{2} < 0 \implies \bar{\mathbf{x}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

- ▶ New column:

$$\left. \begin{array}{l} \mathbf{c}^T \bar{\mathbf{x}} = (1, -3)(5, 0)^T = 5 \\ \mathbf{D}\bar{\mathbf{x}} = (-1, 2)(5, 0)^T = -5 \end{array} \right\} \implies \begin{pmatrix} \frac{5}{-5} \\ 1 \end{pmatrix}$$



# Iteration 2



$$\begin{array}{l|l} \min & -15\lambda_2 + 5\lambda_3 \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \quad \left| \quad \begin{array}{l} \text{Solution:} \\ \text{Dual solution:} \end{array} \quad \begin{array}{l} \boldsymbol{\lambda} = \left(0, \frac{11}{15}, \frac{4}{15}\right)^T \\ \boldsymbol{\pi} = -\frac{4}{3}, q = -\frac{5}{3} \end{array}$$

- ▶ Least reduced cost:

$$\begin{aligned} & \min_{\mathbf{x} \in X} [(\mathbf{c}^T - \boldsymbol{\pi}\mathbf{D})\mathbf{x} - q] \\ = & \min_{\mathbf{x} \in X} \left( [(1, -3) - (-\frac{4}{3})(-1, 2)] \mathbf{x} - (-\frac{5}{3}) \right) \\ & = \min \left\{ -\frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{5}{3} \mid x_1 + x_2 \leq 5; \mathbf{x} \geq \mathbf{0}^2 \right\} = 0 \end{aligned}$$

- ▶ Optimal solution:

$$\boldsymbol{\lambda}^* = \left(0, \frac{11}{15}, \frac{4}{15}\right)^T$$

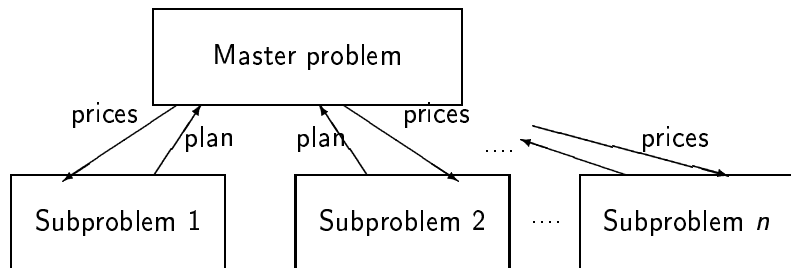
$$\Rightarrow \mathbf{x}^* = (5\lambda_3, 5\lambda_2)^T = \left(\frac{4}{3}, \frac{11}{3}\right)^T; \quad z^* = \frac{4}{3} - 3 \cdot \frac{11}{3} = -9\frac{2}{3}$$

# Block-angular structure

$$\begin{aligned} \max \quad & \mathbf{c}_1^T \mathbf{x}_1 + \mathbf{c}_2^T \mathbf{x}_2 + \cdots + \mathbf{c}_n^T \mathbf{x}_n \\ \text{s.t.} \quad & \mathbf{D}_1 \mathbf{x}_1 + \mathbf{D}_2 \mathbf{x}_2 + \cdots + \mathbf{D}_n \mathbf{x}_n \leq \mathbf{d} \quad | \quad \text{Dual var: } \boldsymbol{\pi} \\ & \mathbf{A}_1 \mathbf{x}_1 \leq \mathbf{b}_1 \quad | \quad \mathbf{x}_1 \in X_1 \\ & \mathbf{A}_2 \mathbf{x}_2 \leq \mathbf{b}_2 \quad | \quad \mathbf{x}_2 \in X_2 \\ & \quad \quad \quad \dots \quad \quad \quad \dots \\ & \mathbf{A}_n \mathbf{x}_n \leq \mathbf{b}_n \quad | \quad \mathbf{x}_n \in X_n \\ & \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \geq \mathbf{0} \\ \\ & X = X_1 \times X_2 \times \dots \times X_n \end{aligned}$$

# DW decomposition as decentralized planning

- ▶ Main office (master problem) sets prizes ( $\pi$ ) for the common resources (complicating constraints)
- ▶ Departments (subproblems) suggest (production) plans ( $\mathbf{D}_j \bar{\mathbf{x}}_j^P$ ) based on given prices
- ▶ Main office mixes suggested plans optimally; sets new prices
- ▶ The procedure is repeated



# Find feasible solutions (right-hand side allocation)

- ▶ Let  $\bar{\lambda}_p^j$ ,  $p \in \mathcal{P}$ , and  $\bar{\mu}_r^j$ ,  $r \in \mathcal{R}$ ,  $j = 1, \dots, n$ , be a feasible and (almost) optimal solution to the master problem
- ▶ A good feasible  $\mathbf{x}$ -solution is then given by (for all  $j$ ):

$$\text{maximize } \mathbf{c}_j^T \mathbf{x}_j$$

$$\text{subject to } \mathbf{D}_j \mathbf{x}_j \leq \sum_{p \in \mathcal{P}} \bar{\lambda}_p^j (\mathbf{D}_j \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}} \bar{\mu}_r^j (\mathbf{D}_j \tilde{\mathbf{x}}_j^r)$$

$$\mathbf{A}_j \mathbf{x}_j \leq \mathbf{b}_j$$

$$\mathbf{x}_j \geq \mathbf{0} \quad [X_j = \{\mathbf{x}_j \geq \mathbf{0} \mid \mathbf{A}_j \mathbf{x}_j \leq \mathbf{b}_j\}]$$

$$\text{since then } \sum_{j=1}^n \mathbf{D}_j \mathbf{x}_j \leq \sum_{j=1}^n \mathbf{D}_j \underbrace{\left( \sum_{p \in \mathcal{P}} \bar{\lambda}_p^j \bar{\mathbf{x}}_j^p + \sum_{r \in \mathcal{R}} \bar{\mu}_r^j \tilde{\mathbf{x}}_j^r \right)}_{\mathbf{x}_j \in X_j} \leq \mathbf{d}$$

# Upper bound on the optimal objective value

$$\begin{aligned} z^* = \min \quad & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{c}^T \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{c}^T \tilde{\mathbf{x}}^r) \\ \text{s.t.} \quad & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{A} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} \quad | \quad \pi \\ & \sum_{p \in \mathcal{P}} \lambda_p = 1 \quad | \quad q \\ & \lambda_p, \mu_r \geq 0, \quad p \in \mathcal{P}, r \in \mathcal{R} \end{aligned}$$

$$\begin{aligned} z^* \leq \bar{z} = \mathbf{d}^T \bar{\pi} + \bar{q} = \max_{(\pi, q)} \mathbf{d}^T \pi + q \\ \text{s.t.} \quad & (\mathbf{D} \bar{\mathbf{x}}^p)^T \pi + q \leq (\mathbf{c}^T \bar{\mathbf{x}}^p), \quad p \in \bar{\mathcal{P}} \\ & (\mathbf{D} \tilde{\mathbf{x}}^r)^T \pi \leq (\mathbf{c}^T \tilde{\mathbf{x}}^r), \quad r \in \bar{\mathcal{R}} \end{aligned}$$

## Lower bound on the optimal objective value

- ▶ Let  $\lambda_p^*$ ,  $p \in \mathcal{P}$ , and  $\mu_r^*$ ,  $r \in \mathcal{R}$ , be optimal in the complete master problem, and  $(\bar{\pi}, \bar{q})$  an optimal dual solution for the columns in  $\bar{\mathcal{P}}$  and  $\bar{\mathcal{R}}$ .
- ▶ Multiply the right-hand side of the primal ( $\mathbf{d}$  resp. 1) by  $\bar{\pi}$  resp.  $\bar{q}$

$\implies$

$$\begin{aligned} 0 &\geq z^* - \bar{z} = z^* - \mathbf{b}^T \bar{\pi} - 1 \cdot \bar{q} \\ &= \sum_{p \in \mathcal{P}} \lambda_p^* [(\mathbf{c}^T \bar{\mathbf{x}}^p) - (\mathbf{D} \bar{\mathbf{x}}^p)^T \bar{\pi} - \bar{q}] + \sum_{r \in \mathcal{R}} \mu_r^* [(\mathbf{c}^T \tilde{\mathbf{x}}^r) - (\mathbf{D} \tilde{\mathbf{x}}^r)^T \bar{\pi}] \\ &\geq \min_{p \in \mathcal{P}} [(\mathbf{c}^T \bar{\mathbf{x}}^p) - (\mathbf{D} \bar{\mathbf{x}}^p)^T \bar{\pi} - \bar{q}] + \sum_{r \in \mathcal{R}} \mu_r^* \min_{s \in \mathcal{R}} [(\mathbf{c}^T \tilde{\mathbf{x}}^s) - (\mathbf{D} \tilde{\mathbf{x}}^s)^T \bar{\pi}] \end{aligned}$$

# Lower bound on the optimal objective value

- ▶ If the subproblem has an unbounded solution no optimistic estimate can be computed in this iteration
- ▶ Otherwise it holds that

$$\min_{s \in \mathcal{R}} [(\mathbf{c}^T \tilde{\mathbf{x}}^s) - (\mathbf{D} \tilde{\mathbf{x}}^s)^T \bar{\boldsymbol{\pi}}] \geq 0$$



$$\begin{aligned} \bar{z} &\geq z^* \geq \bar{z} + \min_{p \in \mathcal{P}} [(\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \bar{\mathbf{x}}^p - \bar{q}] \\ &= \bar{z} + \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} - \bar{q} \\ &= \underline{z} \end{aligned}$$

# Convergence

- ▶ The number of columns generated is finite, since  $X$  is polyhedral
  - ▶ When no more columns are generated, the solution to the last restricted master problem will also solve the original LP
  - ▶ For each new column that is added to the master problem, its optimal objective value will decrease (or be kept constant)
- ⇒ The pessimistic estimate  $\bar{z}_k$  converges monotonically to  $z^*$
- ▶ The optimistic estimate  $\underline{z}_k$  also converges, but perhaps not monotonically
  - ▶ If at iteration  $k$  an optimal solution to the complete master problem is received, then  $\underline{z}_k = \bar{z}_k$  holds
  - ▶ Stopping criterion:  $\bar{z}_k - \underline{z}_k^* \leq \varepsilon$ , where  $\underline{z}_k^* = \max_{s=1, \dots, k} \underline{z}_s$  and  $\varepsilon > 0$



► (IP)

$$\begin{aligned} z_{\text{IP}}^* &= \min \mathbf{c}^T \mathbf{x} \\ \text{s.t. } & \mathbf{D}\mathbf{x} = \mathbf{d} \\ & \mathbf{x} \in X = \{\mathbf{x} \in \mathbb{B}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\} = \{\bar{\mathbf{x}}^p \mid p \in \mathcal{P}\} \end{aligned}$$

► Inner representation (and convexification):

$$\text{conv } X = \left\{ \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p \mid \sum_{p \in \mathcal{P}} \lambda_p = 1; \lambda_p \geq 0, p \in \mathcal{P} \right\}$$

► Let  $c_p = \mathbf{c}^T \bar{\mathbf{x}}^p$  and  $\mathbf{d}_p = \mathbf{D}\bar{\mathbf{x}}^p$ ,  $p \in \mathcal{P}$ . vfill

► (CP)

$$\begin{aligned} z_{\text{IP}}^* = z_{\text{CP}}^* = \min & \sum_{p \in \mathcal{P}} c_p \lambda_p \\ \text{s.t.} & \sum_{p \in \mathcal{P}} \mathbf{d}_p \lambda_p = \mathbf{d} \\ & \sum_{p \in \mathcal{P}} \lambda_p = 1 \\ & \lambda_p \in \{0, 1\}, \quad p \in \mathcal{P} \end{aligned}$$

- A continuous relaxation ((CP<sup>cont</sup>), to  $\lambda_p \geq 0$ ) of (CP) gives the same lower bound as the Lagrangian dual for the constraints  $\mathbf{D}\mathbf{x} = \mathbf{d}$ . ( $z_{\text{LP}}^* \leq z_{\text{CP}}^{\text{cont}} \leq z_{\text{CP}}^*$ )
- The continuous relaxation (LP) of (IP) is never better than any Lagrangian dual bound.

# Restricted master problem

- ▶ Let  $\bar{\mathcal{P}} \subseteq \mathcal{P}$

$$\begin{aligned} (\overline{\text{CP}}) \quad z_{\text{CP}}^* \geq z_{\text{CP}}^{\text{cont}} \leq \bar{z}_{\text{CP}} = \min \quad & \sum_{p \in \bar{\mathcal{P}}} c_p \lambda_p \\ \text{s.t.} \quad & \sum_{p \in \bar{\mathcal{P}}} \mathbf{d}_p \lambda_p = \mathbf{d} \\ & \sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 \quad (*) \\ & \lambda_p \geq 0, \quad p \in \bar{\mathcal{P}} \end{aligned}$$

- ▶ Generate columns  $\begin{pmatrix} c_p \\ \mathbf{d}_p \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$  until an (almost) optimal solution to  $(\text{CP}^{\text{cont}})$ ,  $\hat{\lambda}_p$  ( $p \in \bar{\mathcal{P}}$ ), is found

- ▶  $\hat{\mathbf{x}} = \sum_{p \in \bar{\mathcal{P}}} \hat{\lambda}_p \bar{\mathbf{x}}^p$

# Branching over variable $x_j$ with $0 < \hat{x}_j < 1$

$$\begin{array}{ccc}
 x_j = 0 & \text{or} & x_j = 1 \\
 \Downarrow & & \Downarrow \\
 x_j = \sum_{p \in \bar{P}} \lambda_p \bar{x}_j^p = 0 & & x_j = \sum_{p \in \bar{P}} \lambda_p \bar{x}_j^p = 1
 \end{array}$$

$$\begin{array}{ccc}
 \text{delete col's} & \sum_{p \in \bar{P}: \bar{x}_j^p = 1} \lambda_p = 0 & \sum_{p \in \bar{P}: \bar{x}_j^p = 1} \lambda_p = 1 \text{ replaces } (*) \\
 & \Downarrow & \Downarrow
 \end{array}$$

$$\begin{array}{ccc}
 \text{replaces } (*) & \sum_{p \in \bar{P}: \bar{x}_j^p = 0} \lambda_p = 1 & \sum_{p \in \bar{P}: \bar{x}_j^p = 0} \lambda_p = 0 \text{ delete col's} \\
 & \Downarrow & \Downarrow
 \end{array}$$