Lecture 10: Dantzig–Wolfe decomposition, column generation, and branch–and–price

Ann-Brith Strömberg

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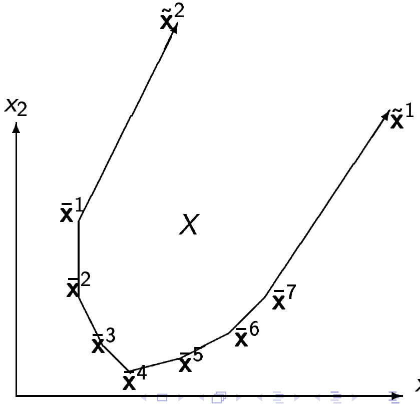
Formulation of LP on column generation form: Dantzig–Wolfe decomposition

- Let $X = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{Ax} = \mathbf{b}\}$ (or $\mathbf{Ax} \leq \mathbf{b}$) be a polyhedron with
- *extreme points* $\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$ and
- *extreme recession directions* $\tilde{\mathbf{x}}^r$, $r \in \mathcal{R}$

Here,

$$\mathcal{P} = \{1, 2, \dots, 7\}$$

$$\mathcal{R} = \{1, 2\}$$



Inner representation of the set X

$$\mathbf{x} \in X \iff \left(\begin{array}{rcl} \mathbf{x} & = & \displaystyle\sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p + \sum_{r \in \mathcal{R}} \mu_r \tilde{\mathbf{x}}^r \\ & & \displaystyle\sum_{p \in \mathcal{P}} \lambda_p = 1 \\ & & \lambda_p \geq 0, \quad p \in \mathcal{P} \\ & & \mu_r \geq 0, \quad r \in \mathcal{R} \end{array} \right)$$

- $\mathbf{x} \in X$ is a *convex* combination of the extreme points plus a *conical* combination of the extreme directions
- Use this *inner representation* of the set X to reformulate an LP according to the *Dantzig-Wolfe decomposition principle*
- Solve by *column generation*

An LP and its correspondingcomplete master problem

$$
\begin{aligned}
\text{(LP1)} \quad z^* = \text{minimum } & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\
\text{subject to } & \mathbf{A}\mathbf{x} = \mathbf{b} && \text{(“simple” constraints)} \\
& \mathbf{D}\mathbf{x} = \mathbf{d} && \text{(complicating constraints)} \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

- Let $X = \{\, \mathbf{x} \geq 0 \mid \mathbf{A}\mathbf{x} = \mathbf{b} \,\}$
- Extreme points $\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$
- Extreme directions $\tilde{\mathbf{x}}^r$, $r \in \mathcal{R}$

$\Longrightarrow$

The complete master problem

$$
\begin{aligned}
\text{(LP2)}\ z^* = \min_{(\lambda,\mu)} \quad & \sum_{p\in\mathcal{P}} \lambda_p(\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p) + \sum_{r\in\mathcal{R}} \mu_r(\mathbf{c}^{\mathrm{T}}\tilde{\mathbf{x}}^r) \\
\text{s.t.} \quad & \sum_{p\in\mathcal{P}} \lambda_p(\mathbf{D}\bar{\mathbf{x}}^p) + \sum_{r\in\mathcal{R}} \mu_r(\mathbf{D}\tilde{\mathbf{x}}^r) = \mathbf{d} && \mid \boldsymbol{\pi} \\
& \sum_{p\in\mathcal{P}} \lambda_p = 1 && \mid q \\
& \lambda_p \geq 0, \quad p \in \mathcal{P} \\
& \mu_r \geq 0, \quad r \in \mathcal{R}
\end{aligned}
$$

- Number of constraints in (LP2) equals “the number of constraints in $\mathbf{Dx} = \mathbf{d}$” $+ 1$
- Number of columns very large (# extreme points/directions of X)

The LP dual of the (restricted) master problem

- Assume that not all extreme points/directions are found: $\bar{\mathcal{P}} \subseteq \mathcal{P}$; $\bar{\mathcal{R}} \subseteq \mathcal{R}$
- The dual of (LP2-R) is given by

$$
\begin{aligned}
\text{(DLP2-R)}\ z^* \leq \max_{(\boldsymbol{\pi}, q)}\ & \mathbf{d}^{\mathrm{T}}\boldsymbol{\pi} + q \\
\text{s.t.}\ & (\mathbf{D}\bar{\mathbf{x}}^p)^{\mathrm{T}}\boldsymbol{\pi} + q \leq (\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p), \quad p \in \bar{\mathcal{P}} \quad | \lambda_p \\
& (\mathbf{D}\tilde{\mathbf{x}}^r)^{\mathrm{T}}\boldsymbol{\pi} \qquad\quad \leq (\mathbf{c}^{\mathrm{T}}\tilde{\mathbf{x}}^r), \quad r \in \bar{\mathcal{R}} \quad | \mu_r
\end{aligned}
$$

  with solution $(\bar{\boldsymbol{\pi}}, \bar{q})$
- Reduced cost for the variable λ_p, $p \in \mathcal{P} \setminus \bar{\mathcal{P}}$:
$$(\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p) - (\mathbf{D}\bar{\mathbf{x}}^p)^{\mathrm{T}}\bar{\boldsymbol{\pi}} - \bar{q} = (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\bar{\boldsymbol{\pi}})^{\mathrm{T}}\bar{\mathbf{x}}^p - \bar{q}$$
- Reduced cost for the variable μ_r, $r \in \mathcal{R} \setminus \bar{\mathcal{R}}$:
$$(\mathbf{c}^{\mathrm{T}}\tilde{\mathbf{x}}^r) - (\mathbf{D}\tilde{\mathbf{x}}^r)^{\mathrm{T}}\bar{\boldsymbol{\pi}} = (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\bar{\boldsymbol{\pi}})^{\mathrm{T}}\tilde{\mathbf{x}}^r$$

Column generation

- The smallest reduced cost is found by solving the subproblem

$$\min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^{\mathrm{T}} \bar{\pi})^{\mathrm{T}} \mathbf{x} \quad \left(\text{alt:} \quad \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^{\mathrm{T}} \bar{\pi})^{\mathrm{T}} \mathbf{x} - \bar{q} \right)$$

- Gives as solution an extreme point, $\bar{\mathbf{x}}^p$, or an extreme direction $\tilde{\mathbf{x}}^r$ (Unbounded solutions can be detected within the simplex method! How?)

$\Longrightarrow$ a new column in (LP2) (if the reduced cost < 0):

- Either $\begin{pmatrix} \mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \mathbf{c}^{\mathrm{T}} \tilde{\mathbf{x}}^r \\ \mathbf{D} \tilde{\mathbf{x}}^r \\ 0 \end{pmatrix}$ enters the problem and improves the solution

Example

(IP)

$$\begin{array}{rrl} z^*_{\mathrm{IP}} & = \min & 2x_1 + 3x_2 + x_3 + 4x_4 \\ & \text{s.t.} & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 \qquad \Big| \ \mathbf{Dx} = \mathbf{d} \\ & & x_1 + x_2 + x_3 + x_4 = 2 \\ & & x_1 \ , \ x_2 \ , \ x_3 \ , \ x_4 \in \{0,1\} \end{array}$$

- $X = \left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\} = \{\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^6\}$
- Optimal solution: $\mathbf{x}^*_{\mathrm{IP}} = (0,1,1,0)^{\mathrm{T}}$
- Optimal value: $z^*_{\mathrm{IP}} = 4$

LP-relaxation

(LP1)

$$\begin{array}{rlr} z^* = \min & 2x_1 + 3x_2 + x_3 + 4x_4 & [\mathbf{c}^{\mathrm{T}}\mathbf{x}] \\ \text{s.t.} & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 & [\mathbf{Dx} = \mathbf{d}] \\ & x_1 + x_2 + x_3 + x_4 = 2 & [\mathbf{x} \in X] \\ 0 \leq & x_1 \;,\; x_2 \;,\; x_3 \;,\; x_4 \leq 1 & [\mathbf{x} \in X] \end{array}$$

- $X = \operatorname{conv}\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\}$

  $= \operatorname{conv}\{\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^6\}$

  $= \left\{ \mathbf{x} \in \mathbb{R}^4 \,\middle|\, \mathbf{x} = \sum_{p=1}^{6} \lambda_p \bar{\mathbf{x}}^p;\; \sum_{p=1}^{6} \lambda_p = 1;\; \lambda_p \geq 0,\; p = 1, \ldots, 6 \right\}$

The complete master problem and the initial columns

(LP2)

$$\begin{aligned} z^* = \min\ & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 + 7\lambda_5 + 5\lambda_6 \\ \text{s.t.}\ & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 + 4\lambda_5 + 5\lambda_6 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 \\ & \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6 \geq 0 \end{aligned}$$

- Initial columns: $\lambda_1, \lambda_2, \lambda_3$

(LP2-R)

$$\begin{aligned} z^* \leq \min\ & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 \\ \text{s.t.}\ & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1,\ \lambda_2,\ \lambda_3 \geq 0 \end{aligned}$$

(DLP2-R)

$$\begin{aligned} z^* \leq \max\ & 5\pi + q \\ \text{s.t.}\ & 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \end{aligned}$$

- Solution: $\bar{\boldsymbol{\lambda}} = (1, 0, 0)^{\mathrm{T}}$, $\bar{\pi} = -2, \quad \bar{q} = 15$

Reduced costs computation

- $$\min_{\mathbf{x}\in X} \left\{ (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\bar{\pi})^{\mathrm{T}}\mathbf{x} - \bar{q} \right\} = \min_{p=1,\dots,6} \left\{ (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\bar{\pi})^{\mathrm{T}}\bar{\mathbf{x}}^p - \bar{q} \right\}$$
$$= \min_{p=1,\dots,6} \left\{ [(2,3,1,4) - (3,2,3,2)\cdot(-2)]\,\bar{\mathbf{x}}^p - 15 \right\}$$
$$= \min\{0,0,1,-1,0,0\} = -1 < 0$$

- New extreme point in (LP1): $\bar{\mathbf{x}}^4 = (0,1,1,0)^{\mathrm{T}}$

- New column in (LP2-R): $\begin{pmatrix} \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^4 \\ \mathbf{D}\bar{\mathbf{x}}^4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

New, extended problem

(LP2-R)

$$
\begin{aligned}
z^* \leq \min\ & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 \\
\text{s.t.}\ & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 = 5 \\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\
& \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0
\end{aligned}
$$

(DLP2-R)

$$
\begin{aligned}
z^* \leq \max\ & 5\pi + q \\
\text{s.t.}\ & 5\pi + q \leq 5 \\
& 6\pi + q \leq 3 \\
& 5\pi + q \leq 6 \\
& 5\pi + q \leq 4
\end{aligned}
$$

- Solution: $\bar{\boldsymbol{\lambda}} = (0,0,0,1)^{\mathrm{T}}$, $\bar{\pi} = -1$, $\bar{q} = 9$
- Reduced costs:
$$\min_{p=1,\ldots,6} \left\{ (5,5,4,6)\, \bar{\mathbf{x}}^p - 9 \right\} = \min\{1,0,2,0,2,1\} = 0$$

Optimal solution to (LP2) and (LP1)

- $\boldsymbol{\lambda}^* = (0,0,0,1,0,0)^{\mathrm{T}}, \qquad \pi^* = -1, \quad q^* = 9$

$\Longrightarrow$ $\mathbf{x}^* = \bar{\mathbf{x}}^4 = (0,1,1,0)^{\mathrm{T}} = \mathbf{x}^*_{\mathrm{IP}}, \qquad z^* = 4 = z^*_{\mathrm{IP}}$

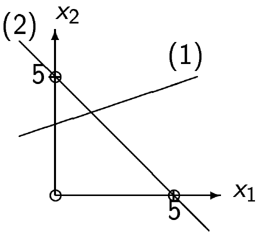
- A coincidence that the solution was integral!
- In general, the solution $\mathbf{x}^*$ to (LP1) may have fractional variable values
- **Solution to (IP)?**
- Need to find an integral solution (not certainly an optimal solution to (IP)) among the columns generated, i.e., solve

$$\min \left\{ (2,3,1,4)\mathbf{x} \,\middle|\, (3,2,3,2)\mathbf{x} = 5,\ \mathbf{x} \in \left\{ \bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \bar{\mathbf{x}}^3, \bar{\mathbf{x}}^4 \right\} \right\}$$

Numerical example of Dantzig-Wolfe decomposition

$$
\begin{array}{lrcrcrl}
\min & x_1 & - & 3x_2 & & & (0) \\
\text{st} & -x_1 & + & 2x_2 & \leq & 6 & (1) \qquad \text{(complicating)} \\
 & x_1 & + & x_2 & \leq & 5 & (2) \\
 & x_1 & , & x_2 & \geq & 0 & (3)
\end{array}
$$

$$
\begin{aligned}
X &= \left\{ \mathbf{x} \in \mathbb{R}^2_+ \,\middle|\, x_1 + x_2 \leq 5 \right\} \\
&= \operatorname{conv} \left\{ (0,0)^{\mathrm{T}}, (0,5)^{\mathrm{T}}, (5,0)^{\mathrm{T}} \right\}
\end{aligned}
$$



Complete DW master problem

- $$\mathbf{x} \in X \iff \left\{ \begin{array}{r} \mathbf{x} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\lambda_3 \\ 5\lambda_2 \end{pmatrix} \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \right\}$$

- $$\begin{array}{llrl} \min & -15\lambda_2 + 5\lambda_3 & & (0) \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 & & (1) \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 & & \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 & & \end{array}$$

- The first restricted master problem is then constructed from the points $(0, 0)^{\mathrm{T}}$ and $(0, 5)^{\mathrm{T}}$ (corresponds to λ_1 and λ_2)

Iteration 1

- $$\begin{array}{llr} \min & -\quad 15\lambda_2 & (0) \\ \text{s.t.} & 10\lambda_2 \le 6 & (1) \\ & \lambda_1 + \lambda_2 = 1 & \\ & \lambda_1, \lambda_2 \ge 0 & \end{array} \quad \Bigg| \quad \begin{array}{ll} \text{Solution:} & \bar{\boldsymbol{\lambda}} = (\frac{2}{5}, \frac{3}{5})^{\mathrm{T}} \\ \text{Dual solution:} & \bar{\pi} = -\frac{3}{2}, \bar{q} = 0 \end{array}$$

- Least reduced cost:

$$\min_{\mathbf{x}\in X} \left[(\mathbf{c}^{\mathrm{T}} - \bar{\pi}\mathbf{D})\mathbf{x} - \bar{q}\right] = \min_{\mathbf{x}\in X} \left(\left[(1,-3) - (-\tfrac{3}{2})(-1,2)\right]\mathbf{x} - 0\right)$$

$$= \min\left\{-\tfrac{1}{2}x_1 \mid x_1 + x_2 \le 5; \mathbf{x} \ge \mathbf{0}^2\right\} = -\tfrac{5}{2} < 0 \Longrightarrow \bar{\mathbf{x}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

- New column:

$$\left.\begin{array}{r} \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}} = (1,-3)(5,0)^{\mathrm{T}} = 5 \\ \mathbf{D}\bar{\mathbf{x}} = (-1,2)(5,0)^{\mathrm{T}} = -5 \end{array}\right\} \Longrightarrow \begin{pmatrix} 5 \\ \hline -5 \\ 1 \end{pmatrix}$$

Iteration 2

- $$\begin{array}{rr} \min & -15\lambda_2 + 5\lambda_3 \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \le 6 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \ge 0 \end{array} \quad \Bigg| \quad \begin{array}{ll} \text{Solution:} & \bar{\boldsymbol{\lambda}} = (0, \frac{11}{15}, \frac{4}{15})^{\mathrm{T}} \\ \text{Dual solution:} & \bar{\pi} = -\frac{4}{3}, \bar{q} = -\frac{5}{3} \end{array}$$
- Least reduced cost:

$$\min_{\mathbf{x}\in X} \left[(\mathbf{c}^{\mathrm{T}} - \bar{\pi}\mathbf{D})\mathbf{x} - \bar{q} \right]$$

$$= \min_{\mathbf{x}\in X} \left(\left[(1,-3) - (-\tfrac{4}{3})(-1,2) \right] \mathbf{x} - (-\tfrac{5}{3}) \right)$$

$$= \min \left\{ -\tfrac{1}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{5}{3} \;\middle|\; x_1 + x_2 \le 5; \mathbf{x} \ge \mathbf{0}^2 \right\} = 0$$

- Optimal solution:

$$\boldsymbol{\lambda}^* = \left(0, \frac{11}{15}, \frac{4}{15}\right)^{\mathrm{T}}$$

$$\Longrightarrow \mathbf{x}^* = (5\lambda_3, 5\lambda_2)^{\mathrm{T}} = (\tfrac{4}{3}, \tfrac{11}{3})^{\mathrm{T}}; \quad z^* = \tfrac{4}{3} - 3 \cdot \tfrac{11}{3} = -9\tfrac{2}{3}$$

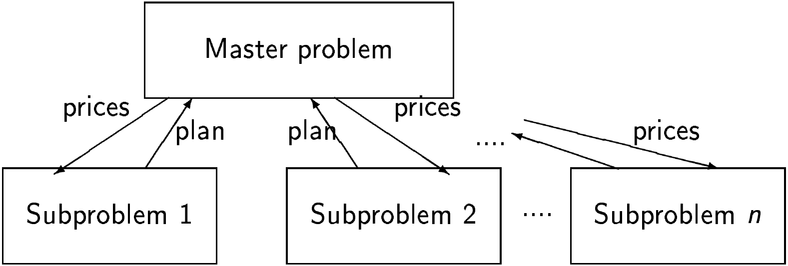
Block-angular structure

$$
\begin{array}{rlll}
\max & \mathbf{c}_1^{\mathrm{T}}\mathbf{x}_1 + \mathbf{c}_2^{\mathrm{T}}\mathbf{x}_2 + \cdots + \mathbf{c}_n^{\mathrm{T}}\mathbf{x}_n & & \\
\text{s.t.} & \mathbf{D}_1\mathbf{x}_1 + \mathbf{D}_2\mathbf{x}_2 + \cdots + \mathbf{D}_n\mathbf{x}_n & \leq \mathbf{d} & \mid \text{Dual var: } \boldsymbol{\pi} \\
& \mathbf{A}_1\mathbf{x}_1 & \leq \mathbf{b}_1 & \mid \mathbf{x}_1 \in X_1 \\
& \qquad\quad \mathbf{A}_2\mathbf{x}_2 & \leq \mathbf{b}_2 & \mid \mathbf{x}_2 \in X_2 \\
& \qquad\qquad\qquad \cdots & \ \ \cdot\ \cdot & \\
& \qquad\qquad\qquad\qquad \mathbf{A}_n\mathbf{x}_n & \leq \mathbf{b}_n & \mid \mathbf{x}_n \in X_n \\
& \qquad\qquad \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n & \geq \mathbf{0} &
\end{array}
$$

$$X = X_1 \times X_2 \times \ldots \times X_n$$

DW decomposition as decentralized planning

- Main office (master problem) sets prizes (π) for the common resources (complicating constraints)
- Departments (subproblems) suggest (production) plans ($\mathbf{D}_j\bar{\mathbf{x}}_j^p$) based on given prices
- Main office mixes suggested plans optimally; sets new prices
- The procedure is repeated



Find feasible solutions (right-hand side allocation)

- Let $\bar{\lambda}_p^j$, $p \in \mathcal{P}$, and $\bar{\mu}_r^j$, $r \in \mathcal{R}$, $j = 1, \ldots, n$, be a feasible and (almost) optimal solution to the master problem
- A good feasible $\mathbf{x}$-solution is then given by (for each j):

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}_j^{\mathrm{T}} \mathbf{x}_j \\ \text{subject to} \quad & \mathbf{D}_j \mathbf{x}_j \leq \sum_{p \in \mathcal{P}} \bar{\lambda}_p^j (\mathbf{D}_j \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}} \bar{\mu}_r^j (\mathbf{D}_j \tilde{\mathbf{x}}_j^r) \\ & \mathbf{A}_j \mathbf{x}_j \leq \mathbf{b}_j \\ & \mathbf{x}_j \geq \mathbf{0} \qquad [X_j = \{\mathbf{x}_j \geq 0 \mid \mathbf{A}_j \mathbf{x}_j \leq \mathbf{b}_j\}] \end{aligned}$$

$$\text{since then } \sum_{j=1}^{n} \mathbf{D}_j \mathbf{x}_j \leq \sum_{j=1}^{n} \mathbf{D}_j \underbrace{\left(\sum_{p \in \mathcal{P}} \bar{\lambda}_p^j \bar{\mathbf{x}}_j^p + \sum_{r \in \mathcal{R}} \bar{\mu}_r^j \tilde{\mathbf{x}}_j^r \right)}_{\mathbf{x}_j \in X_j} \leq \mathbf{d}$$

Upper bound on the optimal objective value

$$
\begin{aligned}
z^* = \min \quad & \sum_{p\in\mathcal{P}} \lambda_p(\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p) + \sum_{r\in\mathcal{R}} \mu_r(\mathbf{c}^{\mathrm{T}}\tilde{\mathbf{x}}^r) \\
\text{s.t.} \quad & \sum_{p\in\mathcal{P}} \lambda_p(\mathbf{A}\bar{\mathbf{x}}^p) + \sum_{r\in\mathcal{R}} \mu_r(\mathbf{D}\tilde{\mathbf{x}}^r) = \mathbf{d} \qquad | \; \boldsymbol{\pi} \\
& \sum_{p\in\mathcal{P}} \lambda_p = 1 \qquad | \; q \\
& \lambda_p, \mu_r \geq 0, \;\; p \in \mathcal{P}, r \in \mathcal{R}
\end{aligned}
$$

$$
\begin{aligned}
z^* \leq \bar{z} = \mathbf{d}^{\mathrm{T}}\bar{\boldsymbol{\pi}} + \bar{q} = \max_{(\boldsymbol{\pi},q)} \quad & \mathbf{d}^{\mathrm{T}}\boldsymbol{\pi} + q \\
\text{s.t.} \quad & (\mathbf{D}\bar{\mathbf{x}}^p)^{\mathrm{T}}\boldsymbol{\pi} + q \leq (\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p), \quad p \in \bar{\mathcal{P}} \\
& (\mathbf{D}\tilde{\mathbf{x}}^r)^{\mathrm{T}}\boldsymbol{\pi} \leq (\mathbf{c}^{\mathrm{T}}\tilde{\mathbf{x}}^r), \quad r \in \bar{\mathcal{R}}
\end{aligned}
$$

Lower bound on the optimal objective value

- Let λ_p^*, $p \in \mathcal{P}$, and μ_r^*, $r \in \mathcal{R}$, be optimal in the complete master problem, and $(\bar{\pi}, \bar{q})$ an optimal dual solution for the columns in $\bar{\mathcal{P}}$ and $\bar{\mathcal{R}}$.
- Multiply the right-hand side of the primal ($\mathbf{d}$ resp. 1) by $\bar{\pi}$ resp. $\bar{q}$

$\Longrightarrow$

$$0 \geq z^* - \bar{z} = z^* - \mathbf{b}^{\mathrm{T}} \bar{\pi} - 1 \cdot \bar{q}$$

$$= \sum_{p \in \mathcal{P}} \lambda_p^* \left[(\mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}^p) - (\mathbf{D}\bar{\mathbf{x}}^p)^{\mathrm{T}} \bar{\pi} - \bar{q} \right] + \sum_{r \in \mathcal{R}} \mu_r^* \left[(\mathbf{c}^{\mathrm{T}} \tilde{\mathbf{x}}^r) - (\mathbf{D}\tilde{\mathbf{x}}^r)^{\mathrm{T}} \bar{\pi} \right]$$

$$\geq \min_{p \in \mathcal{P}} \left[(\mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}^p) - (\mathbf{D}\bar{\mathbf{x}}^p)^{\mathrm{T}} \bar{\pi} - \bar{q} \right] + \sum_{r \in \mathcal{R}} \mu_r^* \min_{s \in \mathcal{R}} \left[(\mathbf{c}^{\mathrm{T}} \tilde{\mathbf{x}}^s) - (\mathbf{D}\tilde{\mathbf{x}}^s)^{\mathrm{T}} \bar{\pi} \right]$$

Lower bound on the optimal objective value

- If the subproblem has an unbounded solution no optimistic estimate can be computed in this iteration
- Otherwise it holds that

$$\min_{s \in \mathcal{R}} \left[(\mathbf{c}^{\mathrm{T}} \tilde{\mathbf{x}}^s) - (\mathbf{D}\tilde{\mathbf{x}}^s)^{\mathrm{T}} \bar{\pi} \right] \geq 0$$

$\Longrightarrow$

$$\begin{aligned} \bar{z} \geq z^* &\geq \bar{z} + \min_{p \in \mathcal{P}} \left[(\mathbf{c} - \mathbf{D}^{\mathrm{T}} \bar{\pi})^{\mathrm{T}} \bar{\mathbf{x}}^p - \bar{q} \right] \\ &= \bar{z} + \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^{\mathrm{T}} \bar{\pi})^{\mathrm{T}} \mathbf{x} - \bar{q} \\ &= \underline{z} \end{aligned}$$

Convergence

- The number of columns generated is finite, since X is polyhedral
- When no more columns are generated, the solution to the last restricted master problem will also solve the original LP
- For each new column that is added to the master problem, its optimal objective value will decrease (or be kept constant)

$\Rightarrow$ The pessimistic estimate $\bar{z}_k$ converges monotonically to z^*

- The optimistic estimate $\underline{z}_k$ also converges, but perhaps not monotonically
- If at iteration k an optimal solution to the complete master problem is received, then $\underline{z}_k = \bar{z}_k$ holds
- Stopping criterion: $\bar{z}_k - \underline{z}_k^* \leq \varepsilon$, where $\underline{z}_k^* = \max_{s=1,\dots,k} \underline{z}_s$ and $\varepsilon > 0$

Branch–and–price for linear 0/1 problems

- (IP)

$$
\begin{aligned}
z^*_{\mathrm{IP}} = \min\ & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\
\text{s.t. } & \mathbf{D}\mathbf{x} = \mathbf{d} \\
& \mathbf{x} \in X = \{\mathbf{x} \in \mathbb{B}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\} = \{\bar{\mathbf{x}}^p \mid p \in \mathcal{P}\}
\end{aligned}
$$

- Inner representation (and convexification):

$$
\operatorname{conv} X = \left\{ \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p \;\middle|\; \sum_{p \in \mathcal{P}} \lambda_p = 1;\ \lambda_p \geq 0,\ \ p \in \mathcal{P} \right\}
$$

- Let $c_p = \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p$ and $\mathbf{d}_p = \mathbf{D}\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$.vfill

Stronger formulation—Master problem

- (CP)

$$
\begin{aligned}
z^*_{\mathrm{IP}} \; = \; z^*_{\mathrm{CP}} = \min \; & \sum_{p \in \mathcal{P}} c_p \lambda_p \\
\text{s.t.} \; & \sum_{p \in \mathcal{P}} \mathbf{d}_p \lambda_p = \mathbf{d} \\
& \sum_{p \in \mathcal{P}} \lambda_p = 1 \\
& \lambda_p \in \{0,1\}, \qquad p \in \mathcal{P}
\end{aligned}
$$

- A continuous relaxation ((CP^{cont}), to $\lambda_p \geq 0$) of (CP) gives the same lower bound as the Lagrangian dual for the constraints $\mathbf{Dx} = \mathbf{d}$. $(z^*_{LP} \leq z^{cont}_{CP} \leq z^*_{CP})$
- The continuous relaxation (LP) of (IP) is never better than any Lagrangian dual bound.

Restricted master problem

- Let $\bar{\mathcal{P}} \subseteq \mathcal{P}$

$$(\overline{\mathrm{CP}}) \qquad z^*_{\mathrm{CP}} \geq z^{cont}_{\mathrm{CP}} \leq \bar{z}_{\mathrm{CP}} = \min \sum_{p \in \bar{\mathcal{P}}} c_p \lambda_p$$

$$\text{s.t.} \sum_{p \in \bar{\mathcal{P}}} \mathbf{d}_p \lambda = \mathbf{d}$$

$$\sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 \qquad (*)$$

$$\lambda_p \geq 0, \qquad p \in \bar{\mathcal{P}}$$

- Generate columns $\begin{pmatrix} c_p \\ \mathbf{d}_p \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ until an (almost) optimal solution to (CP^{cont}), $\widehat{\lambda}_p$ ($p \in \bar{\mathcal{P}}$), is found
- $\widehat{\mathbf{x}} = \sum_{p \in \bar{\mathcal{P}}} \widehat{\lambda}_p \bar{\mathbf{x}}^p$

Branching over variable x_j with $0 < \hat{x}_j < 1$

$$x_j = 0 \qquad \text{or} \qquad x_j = 1$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 0 \qquad x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 1$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\text{delete col's} \quad \sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 1} \lambda_p = 0 \qquad \sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 1} \lambda_p = 1 \quad \text{replaces } (*)$$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$\text{replaces } (*) \quad \sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 0} \lambda_p = 1 \qquad \sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 0} \lambda_p = 0 \quad \text{delete col's}$$