TMA521/MMA510
Optimization, project course
Lecture 2
The solution of a difficult problem—facility location

Ann-Brith Strömberg

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Location of facilities which serve customers

Problem settings

- Potential depot sites: $\mathcal{J} = \{1, \ldots, n\}$ (geographical locations)
- Existing customers: $\mathcal{I} = \{1, \ldots, m\}$ (geographical locations)

  f_j = fixed cost of opening depot (facility) $j \in \mathcal{J}$

  c_{ij} = transportation cost when customer i's demand is fulfilled entirely from depot j ($i \in \mathcal{I}$, $j \in \mathcal{J}$)

Decision problem

- Which depots to open?
- Which depots to serve which customers, and how much?
- **Goal** minimize cost
- **Assumption:** depots have unlimited capacity (to be removed)

Uncapacitated facility location (UFL)

Variables

$$y_j = \begin{cases} 1, & \text{if depot } j \text{ is opened} \\ 0, & \text{otherwise} \end{cases}$$

$$x_{ij} = \left[\begin{array}{l} \text{proportion of customer } i\text{'s demand} \\ \text{to be delivered from depot } j \end{array} \right]$$

Mathematical model

$$\begin{aligned}
z_0^* = \min \quad & \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j && && (0) \\
\text{s.t.} \quad & \sum_{j \in \mathcal{J}} x_{ij} = 1, && i \in \mathcal{I} && (1) \\
& x_{ij} - y_j \leq 0, && i \in \mathcal{I},\ j \in \mathcal{J} && (2) \\
& x_{ij} \in [0,1], && i \in \mathcal{I},\ j \in \mathcal{J} && (3) \\
& y_j \in \{0,1\}, && j \in \mathcal{J} && (4)
\end{aligned}$$

The mathematical model

$$
\begin{array}{rlcll}
z_0^* = \min & \displaystyle\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} c_{ij}x_{ij} + \sum_{j\in\mathcal{J}} f_j y_j & & & (0)\\
\text{s.t.} & \displaystyle\sum_{j\in\mathcal{J}} x_{ij} = 1, & & i\in\mathcal{I} & (1)\\
 & x_{ij} - y_j \leq 0, & & i\in\mathcal{I},\ j\in\mathcal{J} & (2)\\
 & x_{ij} \in [0,1], & & i\in\mathcal{I},\ j\in\mathcal{J} & (3)\\
 & y_j \in \{0,1\}, & & j\in\mathcal{J} & (4)
\end{array}
$$

(0) Minimize cost

(1) Deliver precisely the demand

(2) Deliver from open depots only

(3) $\mathbf{x}_{ij}$ is the *proportion* of the demand of customer i to be delivered from depot j

(4) A depot may *not be partially* opened

Suppose that the depots have limited capacity

- d_i = demand of customer i ($D = \sum_{i \in \mathcal{I}} d_i$)
- b_j = capacity of depot j—if it is opened

  Constraints:

$$\sum_{i \in \mathcal{I}} d_i x_{ij} \leq b_j y_j, \quad j \in \mathcal{J} \quad (5) \qquad (\Rightarrow x_{ij} \leq y_j, \ \forall i,j)$$

$\Rightarrow$ replace (2) (i.e., $x_{ij} \leq y_j$) by (5) $\Rightarrow$

$$\begin{array}{llrcll}
z_0^* = \min & \displaystyle\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j & & & & (0) \\
\text{s.t.} & \displaystyle\sum_{j \in \mathcal{J}} x_{ij} & = & 1, & i \in \mathcal{I} & (1) \\
& \displaystyle\sum_{i \in \mathcal{I}} d_i x_{ij} - b_j y_j & \leq & 0, & j \in \mathcal{J} & (5) \\
& x_{ij} & \in & [0,1], & i \in \mathcal{I}, j \in \mathcal{J} & (3) \\
& y_j & \in & \{0,1\}, & j \in \mathcal{J} & (4)
\end{array}$$

Capacitated facility location (CFL)

$$
\begin{array}{rlllll}
z^* = \min & \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} c_{ij}x_{ij} + \sum_{j\in\mathcal{J}} f_j y_j & & & & (0)\\
\text{s.t.} & \sum_{j\in\mathcal{J}} x_{ij} & = & 1, & i\in\mathcal{I} & (1)\\
& \sum_{i\in\mathcal{I}} d_i x_{ij} - b_j y_j & \le & 0, & j\in\mathcal{J} & (5)\\
& x_{ij} & \in & [0,1], & i\in\mathcal{I},\ j\in\mathcal{J} & (3)\\
& y_j & \in & \{0,1\}, & j\in\mathcal{J} & (4)
\end{array}
$$

Observation: The total capacity of open depots must cover the entire demand $\Longrightarrow$ an additional (redundant) constraint:

$$
(1),(5) \Rightarrow \overbrace{\sum_{j\in\mathcal{J}} b_j y_j}^{\text{capacity}} \ge \sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}} d_i x_{ij} = \sum_{i\in\mathcal{I}} d_i \sum_{j\in\mathcal{J}} x_{ij} = \sum_{i\in\mathcal{I}} d_i \cdot 1 = \overbrace{D}^{\text{demand}} \quad (6)
$$

Add this constraint to the model

Trick – variable splitting

- Replace x_{ij} by w_{ij} in constraint (1) and in "half" the objective
- Let $0 \leq \alpha \leq 1$.
- Add the constraints $x_{ij} = w_{ij}$

$$
\begin{aligned}
z^* = \min \quad & \alpha \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + (1-\alpha) \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} w_{ij} + \sum_{j \in \mathcal{J}} f_j y_j \\
\text{s.t.} \quad & \sum_{j \in \mathcal{J}} w_{ij} = 1, \quad i \in \mathcal{I} && (1) \\
& \sum_{i \in \mathcal{I}} d_i x_{ij} - b_j y_j \leq 0, \quad j \in \mathcal{J} && (5) \\
& \sum_{j \in \mathcal{J}} b_j y_j \geq D, && (6) \\
& w_{ij} - x_{ij} = 0, \quad i \in \mathcal{I},\ j \in \mathcal{J} && (7) \\
& x_{ij} \in [0,1], \quad i \in \mathcal{I},\ j \in \mathcal{J} && (3) \\
& w_{ij} \geq 0, \quad i \in \mathcal{I},\ j \in \mathcal{J} && (8) \\
& y_j \in \{0,1\}, \quad j \in \mathcal{J} && (4)
\end{aligned}
$$

Lagrangian relaxation

- The constraints (7) tie together the variables $(\mathbf{x}, \mathbf{y})$ and $\mathbf{w}$
- Lagrangian relax these with multipliers λ_{ij}

$\Longrightarrow$ Lagrange function

$$L(\mathbf{x}, \mathbf{w}, \mathbf{y}, \boldsymbol{\lambda}) =$$

$$= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left[\alpha c_{ij} x_{ij} + (1-\alpha) c_{ij} w_{ij} + \overbrace{\lambda_{ij}(w_{ij} - x_{ij})}^{\text{penalty}} \right] + \sum_{j \in \mathcal{J}} f_j y_j$$

$$= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} (\alpha c_{ij} - \lambda_{ij}) x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left[(1-\alpha) c_{ij} + \lambda_{ij} \right] w_{ij}$$

- For a fixed value of $\boldsymbol{\lambda}$:
  Minimize the Lagrange function under the constraints (1), (5), (6), (3), (8) & (4)
- Separates into one problem in $(\mathbf{x}, \mathbf{y})$ and $|\mathcal{I}|$ problems in $\mathbf{w}$

The subproblem in x and y (for a fixed value of λ)

$$
\begin{aligned}
q_{\mathbf{xy}}(\boldsymbol{\lambda}) = \min_{x,y} \quad & \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} [\alpha c_{ij} - \lambda_{ij}]\, x_{ij} \quad + \quad \sum_{j\in\mathcal{J}} f_j y_j \\
\text{s.t.} \quad & \sum_{j\in\mathcal{J}} b_j y_j \geq D, & (6) \\
& \sum_{i\in\mathcal{I}} d_i x_{ij} \leq b_j y_j, \quad j\in\mathcal{J} & (5) \\
& x_{ij} \in [0,1], \quad i\in\mathcal{I},\ j\in\mathcal{J} & (3) \\
& y_j \in \{0,1\}, \quad j\in\mathcal{J} & (4)
\end{aligned}
$$

- This problem can be further decomposed through the following observation
- For every **y**-solution (such that $\sum_{j\in\mathcal{J}} b_j y_j \geq D$ holds) we have the following:
  - If $y_j = 0$ then $x_{ij} = 0$, $i \in \mathcal{I}$ must hold
  - If $y_j = 1$ then $\sum_{i\in\mathcal{I}} d_i x_{ij} \leq b_j$ and $x_{ij} \in [0,1]$ must hold

The value/cost of opening depot j, i.e., letting $y_j = 1$ (in the $(\mathbf{x}, \mathbf{y})$-subproblem)

- $|\mathcal{J}|$ continuous knapsack problems (easy to solve)

$$[\text{CKSP}_j] \qquad v_j(\boldsymbol{\lambda}) = f_j + \min_{\mathbf{x}} \sum_{i \in \mathcal{I}} [\alpha c_{ij} - \lambda_{ij}] x_{ij}$$
$$\text{s.t.} \quad \sum_{i \in \mathcal{I}} d_i x_{ij} \leq b_j$$
$$x_{ij} \in [0,1], \quad i \in \mathcal{I}$$

- Then, decide which depots to open (for a certain value of $\boldsymbol{\lambda}$)
- Projection onto the $\mathbf{y}$-space (one 0/1 knapsack problem)

$$[0/1\text{-KSP}] \qquad q_{\mathbf{xy}}(\boldsymbol{\lambda}) = \min_{\mathbf{y}} \sum_{j \in \mathcal{J}} v_j(\boldsymbol{\lambda}) \cdot y_j$$
$$\text{s.t.} \quad \sum_{j \in \mathcal{J}} b_j y_j \geq D,$$
$$y_j \in \{0,1\}, \quad j \in \mathcal{J}$$

Solving the continuous knapsack problems [CKSP$_j$]

Greedy algorithm

- Sort the values $\frac{\alpha c_{ij} - \lambda_{ij}}{d_i} < 0$, $i \in \mathcal{I}$, in increasing order
  $\Rightarrow$ indices $\{i_1, i_2, \dots, i_p\}$, where $p \leq m = |\mathcal{I}|$
- $x_{ij} := 0$, $i \in \mathcal{I}$, $k := 0$

  repeat

  $$k := k + 1$$
  $$x_{i_k j} := \min \left\{ 1; \left(b_j - \sum_{s=1}^{k-1} d_{i_s} x_{i_s j} \right) / d_{i_k} \right\}$$

  until $\sum_{s=1}^{k} d_i x_{i_s j} = b_j$ or $k = p$
- The solution fulfills $\sum_{i \in \mathcal{I}} d_i x_{ij} \leq b_j$ and $x_{ij} \in [0, 1]$, $i \in \mathcal{I}$

$\Rightarrow$ **The value/cost of opening depot j**

- $$v_j(\boldsymbol{\lambda}) = f_j + \sum_{k=1}^{p} \sum_{j \in \mathcal{J}} [\alpha c_{i_k j} - \lambda_{i_k j}] \, x_{i_k j}$$

Solving the $0/1$ knapsack problems [0/1-KSP]

$$
\begin{aligned}
q_{\mathbf{xy}}(\boldsymbol{\lambda}) = \min_{\mathbf{y}} \quad & \sum_{j \in \mathcal{J}} v_j(\boldsymbol{\lambda}) \cdot y_j \\
\text{s.t.} \quad & \sum_{j \in \mathcal{J}} b_j y_j \geq D, \\
& y_j \in \{0, 1\}, \quad j \in \mathcal{J}
\end{aligned}
$$

where $v_j(\boldsymbol{\lambda}) = f_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} [\alpha c_{ij} - \lambda_{ij}] x_{ij}$
and x_{ij}, $i \in \mathcal{I}$, $j \in \mathcal{J}$, are computed by the greedy algorithm

- 0/1-KSP *cannot* be solved in polynomial time
- Solve using Dynamic Programming or Branch & Bound (CPLEX)

Summary of the solution of the $(\mathbf{x}, \mathbf{y})$-problem

For a fixed value of the penalty $\boldsymbol{\lambda}$

- Solve $|\mathcal{J}|$ continuous knapsack problems

$\Rightarrow$ Solution x_{ij}, $i \in \mathcal{I}$, $j \in \mathcal{J}$,

$\Rightarrow$ The value of opening depot j: $v_j(\boldsymbol{\lambda})$, $j \in \mathcal{J}$

- Solve a 0/1-knapsack problem

$\Rightarrow$ $y_j(\boldsymbol{\lambda}) \in \{0, 1\}$, $j \in \mathcal{J}$

  - If $y_j(\boldsymbol{\lambda}) = 0 \Rightarrow x_{ij}(\boldsymbol{\lambda}) = 0$, $i \in \mathcal{I}$
  - If $y_j(\boldsymbol{\lambda}) = 1 \Rightarrow x_{ij}(\boldsymbol{\lambda}) = x_{ij}$ by the above, $i \in \mathcal{I}$,

- Solution $(\mathbf{x}(\boldsymbol{\lambda}), \mathbf{y}(\boldsymbol{\lambda}))$

The subproblem in w (for a fixed value of λ)

$|\mathcal{I}|$ semi-assignment problems (SAP)

$$q_{\mathbf{w}}(\boldsymbol{\lambda}) = \sum_{i \in \mathcal{I}} \left[\begin{array}{ll} \min\limits_{\mathbf{w}} & \sum\limits_{j \in \mathcal{J}} [(1-\alpha)c_{ij} + \lambda_{ij}] \, w_{ij} \\ \text{s.t.} & \sum\limits_{j \in \mathcal{J}} w_{ij} = 1, \quad w_{ij} \geq 0, \quad j \in \mathcal{J} \end{array} \right]$$

Solving semi-assignment problem $i \in \mathcal{I}$

- $\ell_i := \arg\min\limits_{j \in \mathcal{J}} \{(1-\alpha)c_{ij} + \lambda_{ij}\}$
- $w_{i\ell_i}(\boldsymbol{\lambda}) := 1, \; w_{ij}(\boldsymbol{\lambda}) := 0, \; j \neq \ell_i$

The value of the relaxed problem for a fixed value of λ

$$q(\boldsymbol{\lambda}) = \underbrace{q_{\mathbf{xy}}(\boldsymbol{\lambda})}_{\text{difficult}} + \underbrace{q_{\mathbf{w}}(\boldsymbol{\lambda})}_{\text{simple}}$$

- Can show that $q(\boldsymbol{\lambda}) \leq z^*$ for all $\boldsymbol{\lambda} \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{J}|}$ (weak duality)
- λ_{ij} is the penalty for violating the constraint $w_{ij} = x_{ij}$
- Find best *underestimate* of $z^* \iff$ find optimal values for the penalties λ_{ij}
- That is $\boxed{q^* := \max_{\boldsymbol{\lambda} \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{J}|}} q(\boldsymbol{\lambda}) \leq z^*}$
- Most often $\boxed{q^* < z^*}$ (not strong duality)

How to find better values for λ_{ij}?

Penalty: $\min \quad \dots \quad + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \lambda_{ij}(w_{ij} - x_{ij})$

- If $w_{ij}(\boldsymbol{\lambda}) > x_{ij}(\boldsymbol{\lambda}) \Rightarrow$ Increase the value of λ_{ij} (higher penalty for violating the constraint)
- If $w_{ij}(\boldsymbol{\lambda}) < x_{ij}(\boldsymbol{\lambda}) \Rightarrow$ Decrease the value of λ_{ij} (higher penalty for violating the constraint)
- **Iterative method** (subgradient algorithm) to find optimal penalties $\boldsymbol{\lambda}^*$ ($\Rightarrow$ underestimate of $q^* \leq z^*$)

$$\lambda_{ij}^{t+1} = \lambda_{ij}^{t} + \rho_t \left[w_{ij}(\boldsymbol{\lambda}^t) - x_{ij}(\boldsymbol{\lambda}^t) \right], \qquad t = 0, 1, \dots$$

  where $\rho_t > 0$ is a step length, decreasing with t
- Use **feasibility heuristic** from every $\left[\mathbf{x}(\boldsymbol{\lambda}^t), \mathbf{w}(\boldsymbol{\lambda}^t), \mathbf{y}(\boldsymbol{\lambda}^t) \right]$ to yield a **feasible solution** to CFL ($\Rightarrow$ overestimate of z^*)
- E.g., open more depots, send only from open depots, $\mathbf{x} := \mathbf{w}, \dots$

Example: $|\mathcal{I}| = 4$, $|\mathcal{J}| = 3$, $\alpha = \frac{1}{2}$

$$(c_{ij}) = \begin{pmatrix} 6 & 2 & 4 \\ 2 & 8 & 4 \\ 16 & 2 & 6 \\ 10 & 12 & 4 \end{pmatrix}, (f_j) = \begin{pmatrix} 11 \\ 16 \\ 21 \end{pmatrix}, (d_i) = \begin{pmatrix} 6 \\ 4 \\ 8 \\ 5 \end{pmatrix}, (b_j) = \begin{pmatrix} 12 \\ 10 \\ 13 \end{pmatrix}$$

The 0/1-knapsack problem

$$q_{\mathbf{xy}}(\boldsymbol{\lambda}) = \min \quad \sum_{j=1}^{3} v_j(\boldsymbol{\lambda}) \cdot y_j$$
$$\text{s.t.} \quad 12y_1 + 10y_2 + 13y_3 \geq 23$$
$$\mathbf{y} \in \{0,1\}^3$$

Let $(\lambda_{ij}^t) = \begin{pmatrix} 7 & 0 & 0 \\ 3 & 10 & 2 \\ 5 & 2 & 0 \\ 0 & 7 & 5 \end{pmatrix}$

Observe: $y_3 = 1$ must hold (why?)

$\underbrace{\ldots \Longrightarrow \ldots}_{\text{next page}}$

$$q_{\mathbf{xy}}(\boldsymbol{\lambda}^t) = \min \quad 5y_1 + 8.875y_2 + 18y_3$$
$$\text{s.t.} \quad 12y_1 + 10y_2 + 13y_3 \geq 23, \quad \mathbf{y} \in \{0,1\}^3$$

The value of opening a depot

$$v_1(\boldsymbol{\lambda}^t) = 11 + \quad \min \quad -4x_{11} - 2x_{21} + 3x_{31} + 5x_{41}$$
$$\text{s.t.} \quad 6x_{11} + 4x_{21} + 8x_{31} + 5x_{41} \leq 12, \quad \mathbf{x}_{\cdot 1} \in [0,1]^4$$

$$\Rightarrow \boxed{\text{solution } x_{11} = x_{21} = 1,\ x_{31} = x_{41} = 0, \quad v_1(\boldsymbol{\lambda}^t) = 5}$$

$$v_2(\boldsymbol{\lambda}^t) = 16 + \quad \min \quad x_{12} - 6x_{22} - x_{32} - x_{42}$$
$$\text{s.t.} \quad 6x_{12} + 4x_{22} + 8x_{32} + 5x_{42} \leq 10, \quad \mathbf{x}_{\cdot 2} \in [0,1]^4$$

$$\Rightarrow \boxed{\text{solution } x_{22} = x_{42} = 1,\ x_{32} = \tfrac{1}{8},\ x_{12} = 0, \quad v_2(\boldsymbol{\lambda}^t) = 8.875}$$

$$v_3(\boldsymbol{\lambda}^t) = 21 + \quad \min \quad 2x_{13} + 0x_{23} + 3x_{33} - 3x_{43}$$
$$\text{s.t.} \quad 6x_{13} + 4x_{23} + 8x_{33} + 5x_{43} \leq 13, \quad \mathbf{x}_{\cdot 3} \in [0,1]^4$$

$$\Rightarrow \boxed{\text{solution } x_{23} = x_{43} = 1,\ x_{13} = x_{33} = 0, \quad v_3(\boldsymbol{\lambda}^t) = 18}$$

The solution to the (x, y)-problem for $\lambda = \lambda^t$

- Open depots: $\mathbf{y}(\boldsymbol{\lambda}^t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

- Transport goods from open depots to customers (this solution does *not* fulfil the demand constraints (1) for each customer):
$$\mathbf{x}(\boldsymbol{\lambda}^t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Objective value: $q_{\mathbf{xy}}(\boldsymbol{\lambda}^t) = 5 + 0 + 18 = 23$

The w-problem separates into one problem for each customer i

$$\boxed{q_{\mathbf{w}}(\boldsymbol{\lambda}^t) = \sum_{i=1}^{4} q_{\mathbf{w}}^i(\boldsymbol{\lambda}^t)}$$

where

$$\begin{aligned} q_{\mathbf{w}}^i(\boldsymbol{\lambda}^t) = \min_{w} \quad & \sum_{j=1}^{3} \left[(1-\alpha)c_{ij} + \lambda_{ij}^t\right] w_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{3} w_{ij} = 1, \quad w_{ij} \geq 0, \; j = 1, 2, 3 \end{aligned}$$

and $1 - \alpha = \frac{1}{2}$

The solution to the w-problem

$$q_{\mathbf{w}}^1(\boldsymbol{\lambda}^t) = \min \quad 10w_{11} + w_{12} + 2w_{13}$$
$$\text{s.t.} \quad w_{11} + w_{12} + w_{13} = 1, \quad w_{1j} \geq 0, \ j = 1,2,3$$

$$\Rightarrow \text{solution } w_{12}(\boldsymbol{\lambda}^t) = 1, \ w_{11}(\boldsymbol{\lambda}^t) = w_{13}(\boldsymbol{\lambda}^t) = 0, \ q_{\mathbf{w}}^1(\boldsymbol{\lambda}^t) = 1$$

$$q_{\mathbf{w}}^2(\boldsymbol{\lambda}^t) = \min \quad 4w_{21} + 14w_{22} + 4w_{23}$$
$$\text{s.t.} \quad w_{21} + w_{22} + w_{23} = 1, \quad w_{2j} \geq 0, \ j = 1,2,3$$

$$\Rightarrow \text{solution } w_{21}(\boldsymbol{\lambda}^t) = 1, \ w_{22}(\boldsymbol{\lambda}^t) = w_{23}(\boldsymbol{\lambda}^t) = 0, \ q_{\mathbf{w}}^2(\boldsymbol{\lambda}^t) = 4$$

$$q_{\mathbf{w}}^3(\boldsymbol{\lambda}^t) = \min \quad 13w_{31} + 3w_{32} + 3w_{33}$$
$$\text{s.t.} \quad w_{31} + w_{32} + w_{33} = 1, \quad w_{3j} \geq 0, \ j = 1,2,3$$

$$\Rightarrow \text{solution } w_{32}(\boldsymbol{\lambda}^t) = w_{33}(\boldsymbol{\lambda}^t) = \tfrac{1}{2}, \ w_{31}(\boldsymbol{\lambda}^t) = 0, \ q_{\mathbf{w}}^3(\boldsymbol{\lambda}^t) = 3$$

$$q_{\mathbf{w}}^4(\boldsymbol{\lambda}^t) = \min \quad 5w_{41} + 13w_{42} + 7w_{43}$$
$$\text{s.t.} \quad w_{41} + w_{42} + w_{43} = 1, \quad w_{4j} \geq 0, \ j = 1,2,3$$

$$\Rightarrow \text{solution } w_{41}(\boldsymbol{\lambda}^t) = 1, \ w_{42}(\boldsymbol{\lambda}^t) = w_{43}(\boldsymbol{\lambda}^t) = 0, \ q_{\mathbf{w}}^4(\boldsymbol{\lambda}^t) = 5$$

The solution to the (x, y)- and w-problems

- Send the right amout of goods to each customer (this solution presumes that *all* depots are opened):

$$\mathbf{w}(\boldsymbol{\lambda}^t) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

- Objective value: $q_{\mathbf{w}}(\boldsymbol{\lambda}^t) = 13$
- Total objective value $q(\boldsymbol{\lambda}^t) = q_{\mathbf{xy}}(\boldsymbol{\lambda}^t) + q_{\mathbf{w}}(\boldsymbol{\lambda}^t) = 35$
- Lower bound on the optimal objective value: $z^* \geq 35$

Compute a new λ-vector (here, the steplength $\rho_t = 8$)

$$\begin{aligned} \boldsymbol{\lambda}^{t+1} &= \boldsymbol{\lambda}^t + \rho_t \left[\mathbf{w}(\boldsymbol{\lambda}^t) - \mathbf{x}(\boldsymbol{\lambda}^t)\right] \\ &= \begin{pmatrix} 7-\rho_t & \rho_t & 0 \\ 3 & 10 & 2-\rho_t \\ 5 & 2+\frac{\rho_t}{2} & \frac{\rho_t}{2} \\ \rho_t & 7 & 5-\rho_t \end{pmatrix} = \begin{pmatrix} -1 & 8 & 0 \\ 3 & 10 & -6 \\ 5 & 6 & 4 \\ 8 & 7 & -3 \end{pmatrix} \end{aligned}$$

Feasible solution $\Leftrightarrow \mathbf{x}(\boldsymbol{\lambda}^t) = \mathbf{w}(\boldsymbol{\lambda}^t)$? If not $\Rightarrow$ Feasibility heuristic

- Open the depots given by $\mathbf{y}(\boldsymbol{\lambda}^t) \Rightarrow \mathbf{y}^H = \mathbf{y}(\boldsymbol{\lambda}^t) = (1, 0, 1)^{\mathrm{T}}$
- Transport goods only from opened depots:
  $y_j^H = 0 \Rightarrow x_{ij}^H = 0, \; i \in \mathcal{I}$
- Fulfill the demand but do not violate the capacity restrictions

  Let $\mathbf{x}^H = \begin{pmatrix} \frac{1}{6} & 0 & \frac{5}{6} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

  $\Rightarrow z^H = 6 \cdot \frac{1}{6} + 4 \cdot \frac{5}{6} + 2 + 6 + 10 + 11 + 21 = 52 + \frac{1}{3}$

  $\Rightarrow z^* \in [35, 52 + \frac{1}{3}] = [q(\boldsymbol{\lambda}^t), z^H]$

More about the solution method

- Choice of step lengths (ρ_t): Lecture 4 (subgradient optimization, convergence to an optimal value of $\boldsymbol{\lambda}$)
- Feasibility heuristics can be made more or less sophisticated
- There are more ways to Lagrangian relax continuous constraints in an optimization problem
- E.g.: Lagrangian relax (1) *or* (5) (with multipliers $\mu_i \in \mathbb{R}$ and $\nu_j \in \mathbb{R}_+$, respectively) in the original formulation (CFL)

More solution methods for the CFL

- There are also other methods for solving CFL
- E.g., for a fixed value of $\mathbf{y}$, the remaining problem over $\mathbf{x}$ is simple (a transportation problem, network flow)
- An algorithm can be based on only adjusting $\mathbf{y}$, optimizing over $\mathbf{x}$ for each value of $\mathbf{y}$
- The problem is then projected onto the $\mathbf{y}$ variable space
- This is the Benders' decomposition (later in the course)