

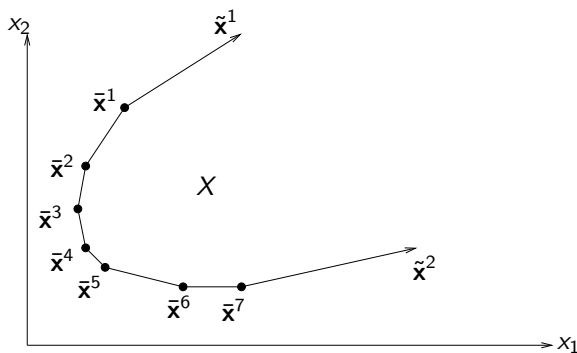
TMA521/MMA510
Optimization, project course
Lecture 8
**Dantzig–Wolfe decomposition, column
generation, and branch–and–price**

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20 September 2010

Formulation of LP on column generation form: Dantzig–Wolfe decomposition

- ▶ Let $X = \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{Ax} = \mathbf{b}\}$ (or $\mathbf{Ax} \leq \mathbf{b}$) be a polyhedron with
- ▶ *extreme points* $\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$ and
- ▶ *extreme recession directions* $\tilde{\mathbf{x}}^r$, $r \in \mathcal{R}$



Inner representation of the set X

$$\mathbf{x} \in X \iff \left(\begin{array}{l} \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p + \sum_{r \in \mathcal{R}} \mu_r \tilde{\mathbf{x}}^r \\ \sum_{p \in \mathcal{P}} \lambda_p = 1 \\ \lambda_p \geq 0, \quad p \in \mathcal{P} \\ \mu_r \geq 0, \quad r \in \mathcal{R} \end{array} \right)$$

- ▶ $\mathbf{x} \in X$ is a *convex* combination of the extreme points plus a *conical* combination of the extreme directions
- ▶ Use this *inner representation* of the set X to reformulate an LP according to the *Dantzig-Wolfe decomposition principle*
- ▶ Solve by *column generation*

An LP and its corresponding complete master problem

$$\begin{aligned} \text{(LP1)} \quad z^* = \text{minimum } & \mathbf{c}^T \mathbf{x} \\ \text{subject to } & \mathbf{Ax} = \mathbf{b} \quad \longleftarrow \text{ ("simple" constraints)} \\ & \mathbf{Dx} = \mathbf{d} \quad \longleftarrow \text{ (complicating constraints)} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- ▶ Let $X = \{ \mathbf{x} \geq \mathbf{0} \mid \mathbf{Ax} = \mathbf{b} \}$
- ▶ Extreme points $\bar{\mathbf{x}}^p, p \in \mathcal{P}$
- ▶ Extreme directions $\tilde{\mathbf{x}}^r, r \in \mathcal{R}$



The complete master problem

$$\begin{aligned} \text{(LP2)} \quad z^* &= \min_{(\lambda, \mu)} \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{c}^T \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{c}^T \tilde{\mathbf{x}}^r) \\ \text{s.t.} \quad &\sum_{p \in \mathcal{P}} \lambda_p (\mathbf{D} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} && | \pi \\ &\sum_{p \in \mathcal{P}} \lambda_p = 1 && | q \\ &\lambda_p \geq 0, \quad p \in \mathcal{P} \\ &\mu_r \geq 0, \quad r \in \mathcal{R} \end{aligned}$$

- ▶ Number of constraints in (LP2) equals “the number of constraints in $\mathbf{D}\mathbf{x} = \mathbf{d}$ ” + 1
- ▶ Number of columns very large ($\#$ extreme points/directions of X)

The LP dual of the (restricted) master problem

- ▶ Assume not all extreme points/directions found: $\bar{\mathcal{P}} \subseteq \mathcal{P}$;
 $\bar{\mathcal{R}} \subseteq \mathcal{R}$
- ▶ The dual of (LP2-R) is given by

$$\begin{aligned} \text{(DLP2-R)} \quad z^* &\leq \max_{(\boldsymbol{\pi}, q)} \mathbf{d}^T \boldsymbol{\pi} + q \\ \text{s.t.} \quad &(\mathbf{D}\bar{\mathbf{x}}^p)^T \boldsymbol{\pi} + q \leq (\mathbf{c}^T \bar{\mathbf{x}}^p), \quad p \in \bar{\mathcal{P}} \quad | \lambda_p \\ &(\mathbf{D}\tilde{\mathbf{x}}^r)^T \boldsymbol{\pi} \leq (\mathbf{c}^T \tilde{\mathbf{x}}^r), \quad r \in \bar{\mathcal{R}} \quad | \mu_r \end{aligned}$$

with solution $(\bar{\boldsymbol{\pi}}, \bar{q})$

- ▶ Reduced cost for the variable λ_p , $p \in \mathcal{P} \setminus \bar{\mathcal{P}}$:
 $(\mathbf{c}^T \bar{\mathbf{x}}^p) - (\mathbf{D}\bar{\mathbf{x}}^p)^T \bar{\boldsymbol{\pi}} - \bar{q} = (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \bar{\mathbf{x}}^p - \bar{q}$
- ▶ Reduced cost for the variable μ_r , $r \in \mathcal{R} \setminus \bar{\mathcal{R}}$:
 $(\mathbf{c}^T \tilde{\mathbf{x}}^r) - (\mathbf{D}\tilde{\mathbf{x}}^r)^T \bar{\boldsymbol{\pi}} = (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \tilde{\mathbf{x}}^r$

Column generation

- ▶ The smallest reduced cost is found by solving the subproblem

$$\min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} \quad \left(\text{alt: } \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} - \bar{q} \right)$$

- ▶ Gives as solution an extreme point, $\bar{\mathbf{x}}^p$, or an extreme direction $\tilde{\mathbf{x}}^r$ (Unbounded solutions can be detected within the simplex method! How?)

⇒ a new column in (LP2) (if the reduced cost < 0):

- ▶ Either $\begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \mathbf{c}^T \tilde{\mathbf{x}}^r \\ \mathbf{D} \tilde{\mathbf{x}}^r \\ 0 \end{pmatrix}$ enters the problem and improves the solution

Example

(IP)

$$\begin{aligned} z_{\text{IP}}^* &= \min 2x_1 + 3x_2 + x_3 + 4x_4 \\ \text{s.t. } &3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 \quad | \quad \mathbf{D}\mathbf{x} = \mathbf{d} \\ &x_1 + x_2 + x_3 + x_4 = 2 \\ &x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

$$\blacktriangleright X = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} = \{\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^6\}$$

\blacktriangleright Optimal solution: $\mathbf{x}_{\text{IP}}^* = (0, 1, 1, 0)^T$

\blacktriangleright Optimal value: $z_{\text{IP}}^* = 4$

(LP1)

$$\begin{aligned}
 z^* = \min \quad & 2x_1 + 3x_2 + x_3 + 4x_4 && [\mathbf{c}^T \mathbf{x}] \\
 \text{s.t.} \quad & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 && [\mathbf{D}\mathbf{x} = \mathbf{d}] \\
 & x_1 + x_2 + x_3 + x_4 = 2 && [\mathbf{x} \in X] \\
 & 0 \leq x_1, x_2, x_3, x_4 \leq 1 && [\mathbf{x} \in X]
 \end{aligned}$$

$$\blacktriangleright X = \text{conv} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$= \text{conv} \{ \bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^6 \}$$

$$= \left\{ \mathbf{x} \in \mathbb{R}^4 \mid \mathbf{x} = \sum_{p=1}^6 \lambda_p \bar{\mathbf{x}}^p; \sum_{p=1}^6 \lambda_p = 1; \lambda_p \geq 0, p = 1, \dots, 6 \right\}$$

The complete master problem and the initial columns

(LP2)

$$\begin{aligned} z^* = \min \quad & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 + 7\lambda_5 + 5\lambda_6 \\ \text{s.t.} \quad & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 + 4\lambda_5 + 5\lambda_6 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 0 \end{aligned}$$

► Initial columns: $\lambda_1, \lambda_2, \lambda_3$

(LP2-R)

$$\begin{aligned} z^* \leq \min \quad & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 \\ \text{s.t.} \quad & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}$$

(DLP2-R)

$$\begin{aligned} z^* \leq \max \quad & 5\pi + q \\ \text{s.t.} \quad & 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \end{aligned}$$

► Solution: $\bar{\lambda} = (1, 0, 0)^T$,

$$\bar{\pi} = -2, \quad \bar{q} = 15$$

Reduced costs computation

$$\begin{aligned}\min_{\mathbf{x} \in X} \left\{ (\mathbf{c} - \mathbf{D}^T \bar{\pi})^T \mathbf{x} - \bar{q} \right\} &= \min_{p=1, \dots, 6} \left\{ (\mathbf{c} - \mathbf{D}^T \bar{\pi})^T \bar{\mathbf{x}}^p - \bar{q} \right\} \\ &= \min_{p=1, \dots, 6} \left\{ [(2, 3, 1, 4) - (3, 2, 3, 2) \cdot (-2)] \bar{\mathbf{x}}^p - 15 \right\} \\ &= \min \{0, 0, 1, -1, 0, 0\} = -1 < 0\end{aligned}$$

▶ New extreme point in (LP1): $\bar{\mathbf{x}}^4 = (0, 1, 1, 0)^T$

▶ New column in (LP2-R): $\begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^4 \\ \mathbf{D} \bar{\mathbf{x}}^4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

New, extended problem

(LP2-R)

$$\begin{aligned} z^* \leq \min & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 \\ \text{s.t.} & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

(DLP2-R)

$$\begin{aligned} z^* \leq \max & 5\pi + q \\ \text{s.t.} & 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \\ & 5\pi + q \leq 4 \end{aligned}$$

► Solution: $\bar{\lambda} = (0, 0, 0, 1)^T$, $\bar{\pi} = -1$, $\bar{q} = 9$

► Reduced costs:

$$\min_{p=1, \dots, 6} \{ (5, 5, 4, 6) \bar{x}^p - 9 \} = \min \{ 1, 0, 2, 0, 2, 1 \} = 0$$

Optimal solution to (LP2) and (LP1)

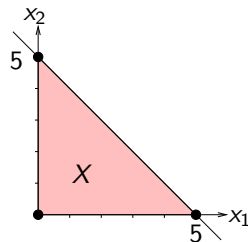
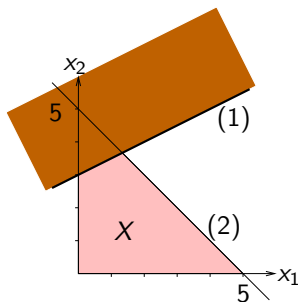
$$\begin{aligned} &\blacktriangleright \boldsymbol{\lambda}^* = (0, 0, 0, 1, 0, 0)^T, \quad \pi^* = -1, \quad q^* = 9 \\ \implies &\mathbf{x}^* = \bar{\mathbf{x}}^4 = (0, 1, 1, 0)^T = \mathbf{x}_{\text{IP}}^*, \quad z^* = 4 = z_{\text{IP}}^* \end{aligned}$$

- ▶ A coincidence that the solution was integral!
- ▶ In general, the solution \mathbf{x}^* to (LP1) may have fractional variable values
- ▶ **Solution to (IP)?**
- ▶ Need to find an integral solution (not certainly an optimal solution to (IP)) among the columns generated, i.e., solve

$$\min \left\{ (2, 3, 1, 4)\mathbf{x} \mid (3, 2, 3, 2)\mathbf{x} = 5, \mathbf{x} \in \{\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \bar{\mathbf{x}}^3, \bar{\mathbf{x}}^4\} \right\}$$

Numerical example of Dantzig-Wolfe decomposition

$$\begin{array}{llllll} \min & x_1 & - & 3x_2 & & (0) \\ \text{st} & -x_1 & + & 2x_2 & \leq & 6 & (1) \quad \leftarrow \quad \text{(complicating)} \\ & x_1 & + & x_2 & \leq & 5 & (2) \\ & x_1 & , & x_2 & \geq & 0 & (3) \end{array}$$



$$X = \{ \mathbf{x} \in \mathbb{R}_+^2 \mid x_1 + x_2 \leq 5 \} = \text{conv} \{ (0,0)^T, (0,5)^T, (5,0)^T \}$$

Complete DW master problem

$$\mathbf{x} \in X \iff \left\{ \begin{array}{l} \mathbf{x} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\lambda_3 \\ 5\lambda_2 \end{pmatrix} \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \right\}$$

$$\begin{array}{ll} \min & -15\lambda_2 + 5\lambda_3 & (0) \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 & (1) \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array}$$

- ▶ The first restricted master problem is then constructed from the points $(0, 0)^T$ and $(0, 5)^T$ (corresponds to λ_1 and λ_2)

Iteration 1



$$\begin{array}{ll} \min & -15\lambda_2 & (0) \\ \text{s.t.} & 10\lambda_2 \leq 6 & (1) \\ & \lambda_1 + \lambda_2 = 1 \\ & \lambda_1, \lambda_2 \geq 0 \end{array} \quad \left| \begin{array}{l} \text{Solution:} \\ \text{Dual solution:} \end{array} \right. \quad \begin{array}{l} \bar{\lambda} = \left(\frac{2}{5}, \frac{3}{5}\right)^T \\ \bar{\pi} = -\frac{3}{2}, \bar{q} = 0 \end{array}$$

- ▶ Smallest reduced cost:

$$\begin{aligned} \min_{\mathbf{x} \in X} [(\mathbf{c}^T - \bar{\pi} \mathbf{D})\mathbf{x} - \bar{q}] &= \min_{\mathbf{x} \in X} ([(1, -3) - (-\frac{3}{2})(-1, 2)] \mathbf{x} - 0) \\ &= \min \left\{ -\frac{1}{2}x_1 \mid x_1 + x_2 \leq 5; \mathbf{x} \geq \mathbf{0}^2 \right\} = -\frac{5}{2} < 0 \implies \bar{\mathbf{x}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \end{aligned}$$

- ▶ New column:

$$\left. \begin{array}{l} \mathbf{c}^T \bar{\mathbf{x}} = (1, -3)(5, 0)^T = 5 \\ \mathbf{D} \bar{\mathbf{x}} = (-1, 2)(5, 0)^T = -5 \end{array} \right\} \implies \begin{pmatrix} -\frac{5}{5} \\ 1 \end{pmatrix}$$

Iteration 2

$$\begin{array}{l|l} \min & -15\lambda_2 + 5\lambda_3 \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \quad \left| \quad \begin{array}{l} \text{Solution:} \quad \bar{\lambda} = \left(0, \frac{11}{15}, \frac{4}{15}\right)^T \\ \text{Dual solution:} \quad \bar{\pi} = -\frac{4}{3}, \bar{q} = -\frac{5}{3} \end{array} \right.$$

- ▶ Smallest reduced cost:

$$\begin{aligned} \min_{\mathbf{x} \in X} [(\mathbf{c}^T - \bar{\pi}\mathbf{D})\mathbf{x} - \bar{q}] &= \min_{\mathbf{x} \in X} \left([(1, -3) - (-\frac{4}{3})(-1, 2)] \mathbf{x} - (-\frac{5}{3}) \right) \\ &= \min \left\{ -\frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{5}{3} \mid x_1 + x_2 \leq 5; \mathbf{x} \geq \mathbf{0}^2 \right\} = 0 \end{aligned}$$

- ▶ Optimal solution: $\lambda^* = \left(0, \frac{11}{15}, \frac{4}{15}\right)^T$

$$\implies \mathbf{x}^* = (5\lambda_3, 5\lambda_2)^T = \left(\frac{4}{3}, \frac{11}{3}\right)^T; \quad z^* = \frac{4}{3} - 3 \cdot \frac{11}{3} = -9\frac{2}{3}$$

Block-angular structure

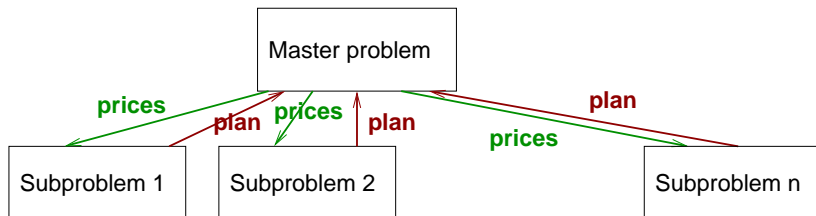
$$\begin{array}{ll} \max & \mathbf{c}_1^T \mathbf{x}_1 + \mathbf{c}_2^T \mathbf{x}_2 + \cdots + \mathbf{c}_n^T \mathbf{x}_n \\ \text{s.t.} & \mathbf{D}_1 \mathbf{x}_1 + \mathbf{D}_2 \mathbf{x}_2 + \cdots + \mathbf{D}_n \mathbf{x}_n \leq \mathbf{d} \quad | \quad \text{Dual var: } \boldsymbol{\pi} \\ & \mathbf{A}_1 \mathbf{x}_1 \leq \mathbf{b}_1 \quad | \quad \mathbf{x}_1 \in X_1 \\ & \mathbf{A}_2 \mathbf{x}_2 \leq \mathbf{b}_2 \quad | \quad \mathbf{x}_2 \in X_2 \\ & \quad \quad \quad \dots \quad \quad \quad \cdot \cdot \\ & \quad \quad \quad \mathbf{A}_n \mathbf{x}_n \leq \mathbf{b}_n \quad | \quad \mathbf{x}_n \in X_n \\ & \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \geq \mathbf{0} \end{array}$$

Cartesian product set:

$$X = X_1 \times X_2 \times \dots \times X_n$$

DW decomposition as decentralized planning

- ▶ Main office (master problem) sets prizes (π) for the common resources (complicating constraints)
- ▶ Departments (subproblems) suggest (production) plans ($(\mathbf{D}_j; \bar{\mathbf{x}}_j^p)$) based on given prices
- ▶ Main office mixes suggested plans optimally; sets new prices
- ▶ The procedure is repeated



Find feasible solutions (right-hand side allocation)

- ▶ Let $\bar{\lambda}_p^j$, $p \in \mathcal{P}$, and $\bar{\mu}_r^j$, $r \in \mathcal{R}$, $j = 1, \dots, n$, be a feasible and (almost) optimal solution to the master problem
- ▶ A good feasible \mathbf{x} -solution is then given by (for each j):

$$\begin{aligned} & \text{maximize } \mathbf{c}_j^T \mathbf{x}_j \\ & \text{subject to } \mathbf{D}_j \mathbf{x}_j \leq \sum_{p \in \mathcal{P}} \bar{\lambda}_p^j (\mathbf{D}_j \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}} \bar{\mu}_r^j (\mathbf{D}_j \tilde{\mathbf{x}}_j^r) \\ & \quad \mathbf{A}_j \mathbf{x}_j \leq \mathbf{b}_j \\ & \quad \mathbf{x}_j \geq \mathbf{0} \quad [X_j = \{\mathbf{x}_j \geq \mathbf{0} \mid \mathbf{A}_j \mathbf{x}_j \leq \mathbf{b}_j\}] \end{aligned}$$

$$\text{since then } \sum_{j=1}^n \mathbf{D}_j \mathbf{x}_j \leq \sum_{j=1}^n \mathbf{D}_j \underbrace{\left(\sum_{p \in \mathcal{P}} \bar{\lambda}_p^j \bar{\mathbf{x}}_j^p + \sum_{r \in \mathcal{R}} \bar{\mu}_r^j \tilde{\mathbf{x}}_j^r \right)}_{\mathbf{x}_j \in X_j} \leq \mathbf{d}$$

Upper bound on the optimal objective value

$$\begin{aligned} z^* = \min \quad & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{c}^T \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{c}^T \tilde{\mathbf{x}}^r) \\ \text{s.t.} \quad & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{A} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} \quad | \quad \boldsymbol{\pi} \\ & \sum_{p \in \mathcal{P}} \lambda_p = 1 \quad | \quad q \\ & \lambda_p, \mu_r \geq 0, \quad p \in \mathcal{P}, r \in \mathcal{R} \end{aligned}$$

$$\begin{aligned} z^* \leq \bar{z} = \mathbf{d}^T \bar{\boldsymbol{\pi}} + \bar{q} = \max_{(\boldsymbol{\pi}, q)} \quad & \mathbf{d}^T \boldsymbol{\pi} + q \\ \text{s.t.} \quad & (\mathbf{D} \bar{\mathbf{x}}^p)^T \boldsymbol{\pi} + q \leq (\mathbf{c}^T \bar{\mathbf{x}}^p), \quad p \in \bar{\mathcal{P}} \\ & (\mathbf{D} \tilde{\mathbf{x}}^r)^T \boldsymbol{\pi} \leq (\mathbf{c}^T \tilde{\mathbf{x}}^r), \quad r \in \bar{\mathcal{R}} \end{aligned}$$

Lower bound on the optimal objective value

- ▶ Let λ_p^* , $p \in \mathcal{P}$, and μ_r^* , $r \in \mathcal{R}$, be optimal in the complete master problem, and $(\bar{\pi}, \bar{q})$ an optimal dual solution for the columns in $\bar{\mathcal{P}}$ and $\bar{\mathcal{R}}$.
- ▶ Multiply the right-hand side of the primal (\mathbf{d} resp. 1) by $\bar{\pi}$ and \bar{q} , resp.

⇒

$$\begin{aligned} 0 &\geq z^* - \bar{z} = z^* - \mathbf{b}^T \bar{\pi} - 1 \cdot \bar{q} \\ &= \sum_{p \in \mathcal{P}} \lambda_p^* [(\mathbf{c}^T \bar{\mathbf{x}}^p) - (\mathbf{D} \bar{\mathbf{x}}^p)^T \bar{\pi} - \bar{q}] + \sum_{r \in \mathcal{R}} \mu_r^* [(\mathbf{c}^T \tilde{\mathbf{x}}^r) - (\mathbf{D} \tilde{\mathbf{x}}^r)^T \bar{\pi}] \\ &\geq \min_{p \in \mathcal{P}} [(\mathbf{c}^T \bar{\mathbf{x}}^p) - (\mathbf{D} \bar{\mathbf{x}}^p)^T \bar{\pi} - \bar{q}] + \sum_{r \in \mathcal{R}} \mu_r^* \min_{s \in \mathcal{R}} [(\mathbf{c}^T \tilde{\mathbf{x}}^s) - (\mathbf{D} \tilde{\mathbf{x}}^s)^T \bar{\pi}] \end{aligned}$$

Lower bound on the optimal objective value

- ▶ If the subproblem has an unbounded solution no optimistic estimate can be computed in this iteration
- ▶ Otherwise it holds that

$$\min_{s \in \mathcal{R}} [(\mathbf{c}^T \tilde{\mathbf{x}}^s) - (\mathbf{D} \tilde{\mathbf{x}}^s)^T \bar{\boldsymbol{\pi}}] \geq 0$$

⇒

$$\begin{aligned} \bar{z} &\geq z^* \geq \bar{z} + \min_{p \in \mathcal{P}} [(\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \bar{\mathbf{x}}^p - \bar{q}] \\ &= \bar{z} + \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} - \bar{q} \\ &= \underline{z} \end{aligned}$$

Convergence

- ▶ The number of columns generated is finite, since X is polyhedral
 - ▶ When no more columns are generated, the solution to the last restricted master problem will also solve the original LP
 - ▶ For each new column that is added to the master problem, its optimal objective value will decrease (or be kept constant)
- ⇒ The pessimistic estimate \bar{z}_k converges monotonically to z^*
- ▶ The optimistic estimate \underline{z}_k also converges, but not monotonically
 - ▶ If at iteration k an optimal solution to the complete master problem is received, then $\underline{z}_k = \bar{z}_k$ holds
 - ▶ Stopping criterion: $\bar{z}_k - \underline{z}_k^* \leq \varepsilon$, where $\underline{z}_k^* = \max_{s=1, \dots, k} \underline{z}_s$ and $\varepsilon > 0$