

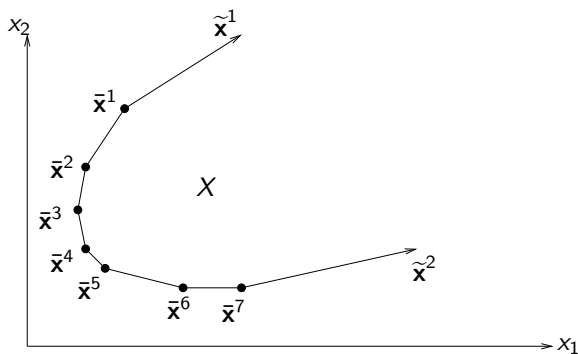
TMA521/MMA510
Optimization, project course
Lecture 8
**Dantzig–Wolfe decomposition, column
generation, and branch–and–price**

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Formulation of LP in a form suitable for column generation: Dantzig–Wolfe decomposition

- ▶ Let $X = \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{Ax} = \mathbf{b}\}$ (or $\mathbf{Ax} \leq \mathbf{b}$) be a polyhedron with
- ▶ *extreme points* $\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$ and
- ▶ *extreme recession directions* $\tilde{\mathbf{x}}^r$, $r \in \mathcal{R}$



Inner representation of the set X

$$\mathbf{x} \in X \iff \left(\begin{array}{l} \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p + \sum_{r \in \mathcal{R}} \mu_r \tilde{\mathbf{x}}^r \\ \sum_{p \in \mathcal{P}} \lambda_p = 1 \\ \lambda_p \geq 0, \quad p \in \mathcal{P} \\ \mu_r \geq 0, \quad r \in \mathcal{R} \end{array} \right)$$

- ▶ $\mathbf{x} \in X$ is a *convex* combination of the extreme points plus a *conical* combination of the extreme directions
- ▶ Use this *inner representation* of the set X to reformulate an LP according to the *Dantzig-Wolfe decomposition principle*
- ▶ Solve by *column generation*

An LP and its corresponding complete master problem

$$\begin{aligned} \text{(LP1)} \quad z^* = \text{minimum } & \mathbf{c}^T \mathbf{x} \\ \text{subject to } & \mathbf{D}\mathbf{x} = \mathbf{d} \quad \longleftarrow \text{ (complicating constraints)} \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \quad \longleftarrow \text{ ("simple" constraints)} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- ▶ Let $X = \{ \mathbf{x} \geq \mathbf{0} \mid \mathbf{A}\mathbf{x} = \mathbf{b} \}$
- ▶ Extreme points $\bar{\mathbf{x}}^p, p \in \mathcal{P}$
- ▶ Extreme directions $\tilde{\mathbf{x}}^r, r \in \mathcal{R}$



The complete master problem

$$\begin{aligned} \text{(LP2) } z^* &= \min_{(\lambda, \mu)} \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{c}^T \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{c}^T \tilde{\mathbf{x}}^r) \\ \text{s.t. } &\sum_{p \in \mathcal{P}} \lambda_p (\mathbf{D} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} && | \pi \\ &\sum_{p \in \mathcal{P}} \lambda_p = 1 && | q \\ &\lambda_p \geq 0, \quad p \in \mathcal{P} \\ &\mu_r \geq 0, \quad r \in \mathcal{R} \end{aligned}$$

- ▶ Number of constraints in (LP2) equals “the number of constraints in $\mathbf{D}\mathbf{x} = \mathbf{d}$ ” + 1
- ▶ Number of columns very large (= # extreme points & directions of X)

The restricted master problem

- ▶ Assume that **not all** extreme points/directions are found:
 $\bar{\mathcal{P}} \subseteq \mathcal{P}; \bar{\mathcal{R}} \subseteq \mathcal{R}$

$$\begin{aligned} \text{(LP2-R)} \quad z^* &= \min_{(\lambda, \mu)} \sum_{p \in \bar{\mathcal{P}}} \lambda_p (\mathbf{c}^T \bar{\mathbf{x}}^p) + \sum_{r \in \bar{\mathcal{R}}} \mu_r (\mathbf{c}^T \tilde{\mathbf{x}}^r) \\ \text{s.t.} \quad & \sum_{p \in \bar{\mathcal{P}}} \lambda_p (\mathbf{D} \bar{\mathbf{x}}^p) + \sum_{r \in \bar{\mathcal{R}}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} && | \pi \\ & \sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 && | q \\ & \lambda_p \geq 0, \quad p \in \bar{\mathcal{P}} \\ & \mu_r \geq 0, \quad r \in \bar{\mathcal{R}} \end{aligned}$$

- ▶ The number of constraints in (LP2) equals “the number of constraints in $\mathbf{D}\mathbf{x} = \mathbf{d}$ ” + 1
- ▶ The number of columns is “smaller”

The LP dual of the (restricted) master problem

- ▶ Assume that **not all** extreme points/directions are found:
 $\bar{\mathcal{P}} \subseteq \mathcal{P}; \bar{\mathcal{R}} \subseteq \mathcal{R}$
- ▶ The dual of (LP2-R) is given by

$$\begin{aligned} \text{(DLP2-R)} \quad z^* &\leq \max_{(\boldsymbol{\pi}, q)} \mathbf{d}^T \boldsymbol{\pi} + q \\ \text{s.t.} \quad &(\mathbf{D}\bar{\mathbf{x}}^p)^T \boldsymbol{\pi} + q \leq (\mathbf{c}^T \bar{\mathbf{x}}^p), \quad p \in \bar{\mathcal{P}} \quad | \lambda_p \\ &(\mathbf{D}\tilde{\mathbf{x}}^r)^T \boldsymbol{\pi} \leq (\mathbf{c}^T \tilde{\mathbf{x}}^r), \quad r \in \bar{\mathcal{R}} \quad | \mu_r \end{aligned}$$

with solution $(\bar{\boldsymbol{\pi}}, \bar{q})$

- ▶ Reduced cost for the variable $\lambda_p, p \in \mathcal{P} \setminus \bar{\mathcal{P}}$:
 $(\mathbf{c}^T \bar{\mathbf{x}}^p) - (\mathbf{D}\bar{\mathbf{x}}^p)^T \bar{\boldsymbol{\pi}} - \bar{q} = (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \bar{\mathbf{x}}^p - \bar{q}$
- ▶ Reduced cost for the variable $\mu_r, r \in \mathcal{R} \setminus \bar{\mathcal{R}}$:
 $(\mathbf{c}^T \tilde{\mathbf{x}}^r) - (\mathbf{D}\tilde{\mathbf{x}}^r)^T \bar{\boldsymbol{\pi}} = (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \tilde{\mathbf{x}}^r$

Column generation

- ▶ The smallest reduced cost is found by solving the column generation subproblem

$$\min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} \quad \left(\text{alt: } \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} - \bar{q} \right)$$

- ▶ Gives as solution an extreme point, $\bar{\mathbf{x}}^p$, or an extreme direction $\tilde{\mathbf{x}}^r$ (Unbounded solutions can be detected within the simplex method! How?)

⇒ a new column in (LP2) (if the reduced cost < 0):

- ▶ Either $\begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \mathbf{c}^T \tilde{\mathbf{x}}^r \\ \mathbf{D} \tilde{\mathbf{x}}^r \\ 0 \end{pmatrix}$ enters the problem and improves the solution

A small example of DW decomposition and column generation

(IP)

$$\begin{aligned} z_{\text{IP}}^* &= \min 2x_1 + 3x_2 + x_3 + 4x_4 \\ \text{s.t. } &3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 \quad | \quad \mathbf{D}\mathbf{x} = \mathbf{d} \\ &x_1 + x_2 + x_3 + x_4 = 2 \\ &x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

$$\blacktriangleright X = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} = \{\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^6\}$$

\blacktriangleright Optimal solution: $\mathbf{x}_{\text{IP}}^* = (0, 1, 1, 0)^T$

\blacktriangleright Optimal value: $z_{\text{IP}}^* = 4$

(LP1)

$$\begin{array}{ll}
 z^* = \min & 2x_1 + 3x_2 + x_3 + 4x_4 & [\mathbf{c}^T \mathbf{x}] \\
 \text{s.t.} & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 & [\mathbf{D}\mathbf{x} = \mathbf{d}] \\
 & x_1 + x_2 + x_3 + x_4 = 2 & [\mathbf{x} \in X] \\
 & 0 \leq x_1, x_2, x_3, x_4 \leq 1 & [\mathbf{x} \in X]
 \end{array}$$

$$\blacktriangleright X = \text{conv} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$= \text{conv} \{ \bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^6 \}$$

$$= \left\{ \mathbf{x} \in \mathbb{R}^4 \mid \mathbf{x} = \sum_{p=1}^6 \lambda_p \bar{\mathbf{x}}^p; \sum_{p=1}^6 \lambda_p = 1; \lambda_p \geq 0, p = 1, \dots, 6 \right\}$$

The complete master problem and the initial columns

(LP2)

$$\begin{aligned} z^* = \min \quad & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 + 7\lambda_5 + 5\lambda_6 \\ \text{s.t.} \quad & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 + 4\lambda_5 + 5\lambda_6 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 0 \end{aligned}$$

► Initial columns: $\lambda_1, \lambda_2, \lambda_3$

(LP2-R)

$$\begin{aligned} z^* \leq \min \quad & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 \\ \text{s.t.} \quad & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}$$

(DLP2-R)

$$\begin{aligned} z^* \leq \max \quad & 5\pi + q \\ \text{s.t.} \quad & 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \end{aligned}$$

► Solution: $\bar{\lambda} = (1, 0, 0)^T$,

$$\bar{\pi} = -2, \quad \bar{q} = 15$$

Reduced costs computation

$$\begin{aligned}\min_{\mathbf{x} \in X} \left\{ (\mathbf{c} - \mathbf{D}^T \bar{\pi})^T \mathbf{x} - \bar{q} \right\} &= \min_{p=1, \dots, 6} \left\{ (\mathbf{c} - \mathbf{D}^T \bar{\pi})^T \bar{\mathbf{x}}^p - \bar{q} \right\} \\ &= \min_{p=1, \dots, 6} \left\{ [(2, 3, 1, 4) - (3, 2, 3, 2) \cdot (-2)] \bar{\mathbf{x}}^p - 15 \right\} \\ &= \min \{0, 0, 1, -1, 0, 0\} = -1 < 0\end{aligned}$$

▶ New extreme point in (LP1): $\bar{\mathbf{x}}^4 = (0, 1, 1, 0)^T$

▶ New column in (LP2-R): $\begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^4 \\ \mathbf{D} \bar{\mathbf{x}}^4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

New, extended problem

(LP2-R)

$$\begin{aligned} z^* \leq \min & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 \\ \text{s.t.} & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

(DLP2-R)

$$\begin{aligned} z^* \leq \max & 5\pi + q \\ \text{s.t.} & 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \\ & 5\pi + q \leq 4 \end{aligned}$$

► Solution: $\bar{\lambda} = (0, 0, 0, 1)^T$, $\bar{\pi} = -1$, $\bar{q} = 9$

► Reduced costs:

$$\min_{p=1, \dots, 6} \{ (5, 5, 4, 6) \bar{x}^p - 9 \} = \min \{ 1, 0, 2, 0, 2, 1 \} = 0$$

Optimal solution to (LP2) and (LP1)

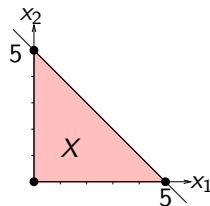
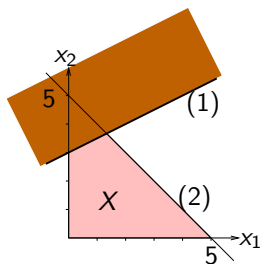
$$\begin{aligned} &\blacktriangleright \boldsymbol{\lambda}^* = (0, 0, 0, 1, 0, 0)^T, \quad \pi^* = -1, \quad q^* = 9 \\ \implies &\mathbf{x}^* = \bar{\mathbf{x}}^4 = (0, 1, 1, 0)^T = \mathbf{x}_{\text{IP}}^*, \quad z^* = 4 = z_{\text{IP}}^* \end{aligned}$$

- ▶ A coincidence that the solution was integral!
- ▶ In general, the solution \mathbf{x}^* to (LP1) may have fractional variable values
- ▶ **Solution to (IP)?**
- ▶ Need to find an integral solution (not certainly an optimal solution to (IP)) among the columns generated, i.e., solve

$$\min \left\{ (2, 3, 1, 4)\mathbf{x} \mid (3, 2, 3, 2)\mathbf{x} = 5, \mathbf{x} \in \{\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \bar{\mathbf{x}}^3, \bar{\mathbf{x}}^4\} \right\}$$

Another numerical example of Dantzig-Wolfe decomposition and column generation

$$\begin{array}{llll} \min & x_1 & - & 3x_2 & (0) \\ \text{st} & -x_1 & + & 2x_2 & \leq 6 & (1) & \longleftarrow & \text{(complicating)} \\ & x_1 & + & x_2 & \leq 5 & (2) \\ & x_1 & , & x_2 & \geq 0 & (3) \end{array}$$



$$X = \{ \mathbf{x} \in \mathbb{R}_+^2 \mid x_1 + x_2 \leq 5 \} = \text{conv} \{ (0,0)^T, (0,5)^T, (5,0)^T \}$$

Complete DW master problem

$$\mathbf{x} \in X \iff \left\{ \begin{array}{l} \mathbf{x} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\lambda_3 \\ 5\lambda_2 \end{pmatrix} \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \right\}$$

$$\begin{array}{ll} \min & -15\lambda_2 + 5\lambda_3 & (0) \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 & (1) \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array}$$

- ▶ The first restricted master problem is then constructed from the points $(0, 0)^T$ and $(0, 5)^T$ (corresponds to λ_1 and λ_2)

Iteration 1



$$\begin{array}{ll} \min & -15\lambda_2 & (0) \\ \text{s.t.} & 10\lambda_2 \leq 6 & (1) \\ & \lambda_1 + \lambda_2 = 1 \\ & \lambda_1, \lambda_2 \geq 0 \end{array} \quad \left| \begin{array}{l} \text{Solution:} \\ \text{Dual solution:} \end{array} \right. \quad \begin{array}{l} \bar{\lambda} = \left(\frac{2}{5}, \frac{3}{5}\right)^T \\ \bar{\pi} = -\frac{3}{2}, \bar{q} = 0 \end{array}$$

- ▶ Smallest reduced cost:

$$\begin{aligned} \min_{\mathbf{x} \in X} [(\mathbf{c}^T - \bar{\pi} \mathbf{D})\mathbf{x} - \bar{q}] &= \min_{\mathbf{x} \in X} ([(1, -3) - (-\frac{3}{2})(-1, 2)] \mathbf{x} - 0) \\ &= \min \left\{ -\frac{1}{2}x_1 \mid x_1 + x_2 \leq 5; \mathbf{x} \geq \mathbf{0}^2 \right\} = -\frac{5}{2} < 0 \implies \bar{\mathbf{x}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \end{aligned}$$

- ▶ New column:

$$\left. \begin{array}{l} \mathbf{c}^T \bar{\mathbf{x}} = (1, -3)(5, 0)^T = 5 \\ \mathbf{D} \bar{\mathbf{x}} = (-1, 2)(5, 0)^T = -5 \end{array} \right\} \implies \begin{pmatrix} -\frac{5}{5} \\ 1 \end{pmatrix}$$

Iteration 2

$$\begin{array}{l|l} \min & -15\lambda_2 + 5\lambda_3 \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \quad \left| \quad \begin{array}{l} \text{Solution:} \quad \bar{\lambda} = \left(0, \frac{11}{15}, \frac{4}{15}\right)^T \\ \text{Dual solution:} \quad \bar{\pi} = -\frac{4}{3}, \bar{q} = -\frac{5}{3} \end{array} \right.$$

- ▶ Smallest reduced cost:

$$\begin{aligned} \min_{\mathbf{x} \in X} [(\mathbf{c}^T - \bar{\pi}\mathbf{D})\mathbf{x} - \bar{q}] &= \min_{\mathbf{x} \in X} \left([(1, -3) - (-\frac{4}{3})(-1, 2)] \mathbf{x} - (-\frac{5}{3}) \right) \\ &= \min \left\{ -\frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{5}{3} \mid x_1 + x_2 \leq 5; \mathbf{x} \geq \mathbf{0}^2 \right\} = 0 \end{aligned}$$

- ▶ Optimal solution: $\lambda^* = \left(0, \frac{11}{15}, \frac{4}{15}\right)^T$

$$\implies \mathbf{x}^* = (5\lambda_3, 5\lambda_2)^T = \left(\frac{4}{3}, \frac{11}{3}\right)^T; \quad z^* = \frac{4}{3} - 3 \cdot \frac{11}{3} = -9\frac{2}{3}$$

DW decomposition applied to an LP with block-angular structure

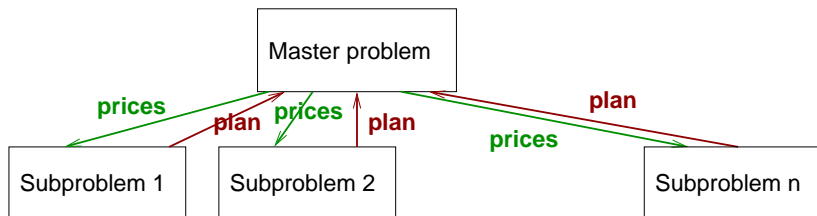
$$\begin{array}{ll} \max & \mathbf{c}_1^T \mathbf{x}_1 + \mathbf{c}_2^T \mathbf{x}_2 + \cdots + \mathbf{c}_n^T \mathbf{x}_n \\ \text{s.t.} & \mathbf{D}_1 \mathbf{x}_1 + \mathbf{D}_2 \mathbf{x}_2 + \cdots + \mathbf{D}_n \mathbf{x}_n \leq \mathbf{d} \quad | \quad \text{Dual var: } \boldsymbol{\pi} \\ & \mathbf{A}_1 \mathbf{x}_1 \leq \mathbf{b}_1 \quad | \quad \mathbf{x}_1 \in X_1 \\ & \mathbf{A}_2 \mathbf{x}_2 \leq \mathbf{b}_2 \quad | \quad \mathbf{x}_2 \in X_2 \\ & \quad \quad \quad \dots \quad \quad \quad \cdot \cdot \\ & \mathbf{A}_n \mathbf{x}_n \leq \mathbf{b}_n \quad | \quad \mathbf{x}_n \in X_n \\ & \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \geq \mathbf{0} \end{array}$$

Cartesian product set:

$$X = X_1 \times X_2 \times \dots \times X_n$$

DW decomposition as decentralized planning

- ▶ The main office (master problem) sets prizes (π) for the common resources (complicating constraints)
- ▶ The departments (subproblems) suggest (production) plans (i.e., columns) ($D_j \bar{x}_j^P$) based on given prices
- ▶ The main office “mixes” the suggested plans (columns) optimally; sets new prices
- ▶ The procedure is repeated



Inner representations of the sets X_j , $j = 1, \dots, n$

- ▶ Let $X_j = \{\mathbf{x}_j \geq \mathbf{0} \mid \mathbf{A}_j \mathbf{x}_j \leq \mathbf{b}_j\}$ and express \mathbf{x}_j as

$$\mathbf{x}_j \in X_j \iff \left(\begin{array}{l} \mathbf{x}_j = \sum_{p \in \mathcal{P}_j} \lambda_{pj} \bar{\mathbf{x}}_j^p + \sum_{r \in \mathcal{R}_j} \mu_{rj} \tilde{\mathbf{x}}_j^r \\ \sum_{p \in \mathcal{P}_j} \lambda_{pj} = 1 \\ \lambda_{pj} \geq 0, \quad p \in \mathcal{P}_j \\ \mu_{rj} \geq 0, \quad r \in \mathcal{R}_j \end{array} \right) \quad j = 1, \dots, n$$

\iff

- ▶ $X_j = \text{conv}\{\bar{\mathbf{x}}_j^p \mid p \in \mathcal{P}_j\} + \text{cone}\{\tilde{\mathbf{x}}_j^r \mid r \in \mathcal{R}_j\}$, $j = 1, \dots, n$

The complete master problem

$$\max_{(\lambda, \mu)} \sum_{j=1}^n \left(\sum_{p \in \mathcal{P}_j} \lambda_{pj} (\mathbf{c}^T \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}_j} \mu_{rj} (\mathbf{c}^T \tilde{\mathbf{x}}_j^r) \right)$$

$$\text{s.t. } \sum_{j=1}^n \left(\sum_{p \in \mathcal{P}_j} \lambda_{pj} (\mathbf{D} \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}_j} \mu_{rj} (\mathbf{D} \tilde{\mathbf{x}}_j^r) \right) \leq \mathbf{d}$$

$$\sum_{p \in \mathcal{P}_j} \lambda_{pj} = 1, \quad j = 1, \dots, n$$

$$\lambda_{pj} \geq 0, \quad p \in \mathcal{P}_j, \quad j = 1, \dots, n$$

$$\mu_{rj} \geq 0, \quad r \in \mathcal{R}_j, \quad j = 1, \dots, n$$

- ▶ The number of constraints in the master problem equals “the number of constraints in $\mathbf{D}\mathbf{x} = \mathbf{d}$ ” + n
- ▶ The restricted master problem is formulated analogously as before (with $\bar{\mathcal{P}}_j \subseteq \mathcal{P}_j$ and $\bar{\mathcal{R}}_j \subseteq \mathcal{R}_j$, $j = 1, \dots, n$)

Use the LP dual of the restricted master problem

- ▶ The LP dual of the restricted master problem

$$\min_{(\boldsymbol{\pi}, \mathbf{q})} \quad \mathbf{d}^T \boldsymbol{\pi} + \sum_{j=1}^n q_j$$

$$\text{s.t. } (\mathbf{D}_j \bar{\mathbf{x}}_j^p)^T \boldsymbol{\pi} + q_j \geq (\mathbf{c}_j^T \bar{\mathbf{x}}_j^p), \quad p \in \bar{\mathcal{P}}_j, j = 1, \dots, n \quad | \lambda_{pj}$$

$$(\mathbf{D}_j \tilde{\mathbf{x}}_j^r)^T \boldsymbol{\pi} \geq (\mathbf{c}_j^T \tilde{\mathbf{x}}_j^r), \quad r \in \tilde{\mathcal{R}}_j, j = 1, \dots, n \quad | \mu_{rj}$$

with solution $\bar{\boldsymbol{\pi}}, \bar{q}_j, j = 1, \dots, n$

- ▶ Generate new columns (maximization \Leftrightarrow reduced cost > 0):

For $j = 1, \dots, n$, solve $\max_{\mathbf{x}_j \in X_j} \left\{ (\mathbf{c}_j - \mathbf{D}_j^T \bar{\boldsymbol{\pi}})^T \mathbf{x}_j - \bar{q}_j \right\}$

\Rightarrow Solution $\bar{\mathbf{x}}_j^p$ or $\tilde{\mathbf{x}}_j^r$

\Rightarrow Column $\begin{pmatrix} \mathbf{c}_j^T \bar{\mathbf{x}}_j^p \\ \mathbf{D}_j^T \bar{\mathbf{x}}_j^p \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \mathbf{c}_j^T \tilde{\mathbf{x}}_j^r \\ \mathbf{D}_j^T \tilde{\mathbf{x}}_j^r \\ 0 \end{pmatrix}$ for each $j \in \{1, \dots, n\}$

Find feasible solutions (right-hand side allocation)

- ▶ Let $\bar{\lambda}_{pj}$, $p \in \mathcal{P}_j$, and $\bar{\mu}_{rj}$, $r \in \mathcal{R}_j$, $j = 1, \dots, n$, be a **feasible** and (almost) optimal solution to the restricted master problem
- ▶ It then holds that

$$\sum_{j=1}^n \mathbf{D}_j \underbrace{\left(\sum_{p \in \mathcal{P}} \bar{\lambda}_{pj} \bar{\mathbf{x}}_j^p + \sum_{r \in \mathcal{R}} \bar{\mu}_{rj} \tilde{\mathbf{x}}_j^r \right)}_{\in X_j} \leq \mathbf{d}$$

- ▶ Therefore, a good feasible \mathbf{x} -solution is given by the solution to the program

$$\begin{aligned} & \text{maximize } \mathbf{c}_j^T \mathbf{x}_j \\ & \text{subject to } \mathbf{D}_j \mathbf{x}_j \leq \sum_{p \in \mathcal{P}} \bar{\lambda}_{pj} (\mathbf{D}_j \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}} \bar{\mu}_{rj} (\mathbf{D}_j \tilde{\mathbf{x}}_j^r) \\ & \quad \mathbf{x}_j \in X_j \end{aligned}$$

for $j = 1, \dots, n$

Upper bound on the optimal objective value for LP1

$$\begin{aligned} z^* = \min \quad & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{c}^T \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{c}^T \tilde{\mathbf{x}}^r) \\ \text{s.t.} \quad & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{A} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} \quad | \quad \boldsymbol{\pi} \\ & \sum_{p \in \mathcal{P}} \lambda_p = 1 \quad | \quad q \\ & \lambda_p, \mu_r \geq 0, \quad p \in \mathcal{P}, r \in \mathcal{R} \end{aligned}$$

$$\begin{aligned} z^* \leq \bar{z} = \mathbf{d}^T \bar{\boldsymbol{\pi}} + \bar{q} = \max_{(\boldsymbol{\pi}, q)} \quad & \mathbf{d}^T \boldsymbol{\pi} + q \\ \text{s.t.} \quad & (\mathbf{D} \bar{\mathbf{x}}^p)^T \boldsymbol{\pi} + q \leq (\mathbf{c}^T \bar{\mathbf{x}}^p), \quad p \in \bar{\mathcal{P}} \\ & (\mathbf{D} \tilde{\mathbf{x}}^r)^T \boldsymbol{\pi} \leq (\mathbf{c}^T \tilde{\mathbf{x}}^r), \quad r \in \bar{\mathcal{R}} \end{aligned}$$

Lower bound on the optimal objective value for LP1

- ▶ Let λ_p^* , $p \in \mathcal{P}$, and μ_r^* , $r \in \mathcal{R}$, be optimal in the complete master problem, and $(\bar{\pi}, \bar{q})$ an optimal dual solution for the columns in $\bar{\mathcal{P}}$ and $\bar{\mathcal{R}}$
- ▶ Multiply the right-hand side of the primal (\mathbf{d} and 1) by $\bar{\pi}$ and \bar{q} , respectively



$$\begin{aligned} 0 &\geq z^* - \bar{z} = z^* - \mathbf{d}^T \bar{\pi} - 1 \cdot \bar{q} \\ &= \sum_{p \in \mathcal{P}} \lambda_p^* [(\mathbf{c}^T \bar{\mathbf{x}}^p) - (\mathbf{D} \bar{\mathbf{x}}^p)^T \bar{\pi} - \bar{q}] + \sum_{r \in \mathcal{R}} \mu_r^* [(\mathbf{c}^T \tilde{\mathbf{x}}^r) - (\mathbf{D} \tilde{\mathbf{x}}^r)^T \bar{\pi}] \\ &\geq \min_{p \in \mathcal{P}} [(\mathbf{c}^T \bar{\mathbf{x}}^p) - (\mathbf{D} \bar{\mathbf{x}}^p)^T \bar{\pi} - \bar{q}] + \sum_{r \in \mathcal{R}} \mu_r^* \min_{s \in \mathcal{R}} [(\mathbf{c}^T \tilde{\mathbf{x}}^s) - (\mathbf{D} \tilde{\mathbf{x}}^s)^T \bar{\pi}] \end{aligned}$$

Lower bound on the optimal objective value for LP1

- ▶ If the subproblem has an unbounded solution no optimistic estimate can be computed in this iteration
- ▶ Otherwise it holds that

$$\min_{s \in \mathcal{R}} [(\mathbf{c}^T \tilde{\mathbf{x}}^s) - (\mathbf{D} \tilde{\mathbf{x}}^s)^T \bar{\boldsymbol{\pi}}] \geq 0$$

⇒

$$\begin{aligned} \bar{z} &\geq z^* \geq \bar{z} + \min_{p \in \mathcal{P}} [(\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \bar{\mathbf{x}}^p - \bar{q}] \\ &= \bar{z} + \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^T \bar{\boldsymbol{\pi}})^T \mathbf{x} - \bar{q} \\ &= \underline{z} \end{aligned}$$

Convergence of the column generation algorithm

- ▶ The number of columns generated is finite, since X is polyhedral
 - ▶ When no more columns are generated, the solution to the last restricted master problem will also solve the original LP
 - ▶ For each new column that is added to the master problem, its optimal objective value will decrease (or be kept constant)
- ⇒ The pessimistic estimate \bar{z}_k converges monotonically to z^*
- ▶ The optimistic estimate \underline{z}_k also converges, but not monotonically
 - ▶ If at iteration k an optimal solution to the complete master problem is received, then $\underline{z}_k = \bar{z}_k$ holds
 - ▶ Stopping criterion: $\bar{z}_k - \underline{z}_k^* \leq \varepsilon$, where $\underline{z}_k^* = \max_{s=1, \dots, k} \underline{z}_s$ and $\varepsilon > 0$

Branch-and-price for linear 0/1 problems

(IP)

$$\begin{aligned} z_{\text{IP}}^* &= \min \mathbf{c}^T \mathbf{x} \\ \text{s.t. } & \mathbf{D}\mathbf{x} = \mathbf{d} \\ & \mathbf{x} \in X = \{\mathbf{x} \in \mathbb{B}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\} = \{\bar{\mathbf{x}}^p \mid p \in \mathcal{P}\} \end{aligned}$$

- ▶ Inner representation (and convexification):

$$\text{conv } X = \left\{ \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p \mid \sum_{p \in \mathcal{P}} \lambda_p = 1; \lambda_p \geq 0, p \in \mathcal{P} \right\}$$

- ▶ Let $c_p = \mathbf{c}^T \bar{\mathbf{x}}^p$ and $\mathbf{d}_p = \mathbf{D}\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$.

Stronger formulation—Master problem

(CP)

$$\begin{aligned} z_{\text{IP}}^* = z_{\text{CP}}^* = \min \quad & \sum_{p \in \mathcal{P}} c_p \lambda_p \\ \text{s.t.} \quad & \sum_{p \in \mathcal{P}} \mathbf{d}_p \lambda_p = \mathbf{d} \\ & \sum_{p \in \mathcal{P}} \lambda_p = 1 \\ & \lambda_p \in \{0, 1\}, \quad p \in \mathcal{P} \end{aligned}$$

- ▶ A continuous relaxation ((CP^{cont}), to $\lambda_p \geq 0$) of (CP) gives the same lower bound as the Lagrangian dual with respect to the constraints $\mathbf{D}\mathbf{x} = \mathbf{d}$ ($z_{\text{LP}}^* \leq z_{\text{CP}}^{\text{cont}} \leq z_{\text{CP}}^*$)
- ▶ The continuous relaxation (LP) of (IP) is never better than any Lagrangian dual bound.

Restricted master problem

- ▶ Let $\bar{\mathcal{P}} \subseteq \mathcal{P}$

$$\begin{aligned} (\bar{\text{CP}}) \quad z_{\text{CP}}^* \geq z_{\text{CP}}^{\text{cont}} \leq \bar{z}_{\text{CP}} = \min \quad & \sum_{p \in \bar{\mathcal{P}}} c_p \lambda_p \\ \text{s.t.} \quad & \sum_{p \in \bar{\mathcal{P}}} \mathbf{d}_p \lambda_p = \mathbf{d} \\ & \sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 \quad (*) \\ & \lambda_p \geq 0, \quad p \in \bar{\mathcal{P}} \end{aligned}$$

- ▶ Generate columns $\begin{pmatrix} c_p \\ \mathbf{d}_p \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ until an (almost) optimal solution to $(\text{CP}^{\text{cont}})$, $\hat{\lambda}_p$ ($p \in \bar{\mathcal{P}}$), is found
- ▶ The corresponding \mathbf{x} -solution: $\hat{\mathbf{x}} = \sum_{p \in \bar{\mathcal{P}}} \hat{\lambda}_p \bar{\mathbf{x}}^p$

Branching over variable x_j with $0 < \hat{x}_j < 1$

$$x_j = 0 \quad \text{or} \quad x_j = 1$$



$$x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 0 \quad x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 1$$



delete col's $\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 1} \lambda_p = 0$

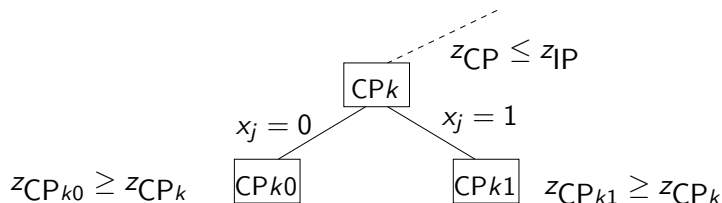
$\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 1} \lambda_p = 1$ replaces (*)



replaces (*) $\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 0} \lambda_p = 1$

$\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 0} \lambda_p = 0$ delete col's

Generate columns in B&B nodes



- ▶ In each node (CP , CP_0 , CP_1 , ...): Generate columns until (almost) optimal (all reduced costs ≥ 0) or verified infeasible
 - ▶ If $\mathbf{x}_{CP_{kl}\dots}^*$ feasible $\implies z_{CP_{kl}\dots}^* \geq z_{IP}^* \implies$ Cut off the branch (k, l, \dots)
- \implies Cut branches (r, s, \dots) with $z_{CP_{rs}\dots}^* \geq z_{CP_{kl}\dots}^*$

The column generation subproblem, reduced costs

- ▶ $\min_{\mathbf{x} \in X^k} (\mathbf{c} - \mathbf{D}^T \hat{\boldsymbol{\pi}}^k)^T \mathbf{x} - \hat{q}^k =: (\mathbf{c} - \mathbf{D}^T \hat{\boldsymbol{\pi}}^k)^T \bar{\mathbf{x}}^P - \hat{q}^k =: \bar{c}(\bar{\mathbf{x}}^P)$
- ▶ $(\hat{\boldsymbol{\pi}}^k, \hat{q}^k)$ is a dual solution to the RMP and $X^k = X \cap \{\mathbf{x} \mid x_j = k\}$, $k \in \{0, 1\}$ (etc. down the tree)
- ▶ If $\bar{c}(\bar{\mathbf{x}}^P) < 0$ then $\begin{pmatrix} \mathbf{c}^T \bar{\mathbf{x}}^P \\ \mathbf{D} \bar{\mathbf{x}}^P \\ 1 \end{pmatrix}$ is a new column in [CPk]
- ▶ Minimization? $\bar{\mathbf{x}}^r$ is good enough if $\bar{c}(\bar{\mathbf{x}}^r) < 0$
- ▶ If $\bar{c}(\bar{\mathbf{x}}^P) \geq 0$ then no more columns are needed to solve [CPk] to optimality.
- ▶ The same columns may be generated in different nodes \implies create “column pool” to check w.r.t. reduced costs \bar{c}

An instance solved by Branch-and-price

$$\begin{aligned} z_{IP}^* = \min \quad & x_1 + 2x_2 = z_{CP}^* \geq z_{CP}^{cont} = z_{LP}^* = \min \quad x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + 2x_2 \geq 1 \qquad \qquad \qquad \text{s.t.} \quad 2x_1 + 2x_2 \geq 1 \\ & x_1, x_2 \in \{0, 1\} \qquad \qquad \qquad \qquad \qquad \qquad 0 \leq x_1, x_2 \leq 1 \end{aligned}$$

$$\text{conv}X = \text{conv} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \lambda_3 + \lambda_4 \\ \lambda_2 + \lambda_4 \end{pmatrix} \mid \sum_{p=1}^4 \lambda_p = 1; \lambda_p \geq 0 \right\}$$

$$\begin{aligned} \text{[CP]} \quad & z_{CP}^{cont} = \min \quad 2\lambda_2 + \lambda_3 + 3\lambda_4 \\ & \text{s.t.} \quad 2\lambda_2 + 2\lambda_3 + 4\lambda_4 \geq 1 \\ & \quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ & \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

Initial columns: λ_1 and λ_3

Choose e.g., $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, that is, the variables λ_1 and λ_3

$$\begin{aligned} z_{CP}^{cont} \leq \min \quad & \lambda_3 & = \quad & \max \quad \pi + q \\ \text{s.t.} \quad & 2\lambda_3 \geq 1 & \text{s.t.} \quad & q \leq 0 \\ & \lambda_1 + \lambda_3 = 1 & & 2\pi + q \leq 1 \\ & \lambda_1, \lambda_3 \geq 0 & & \pi \geq 0 \end{aligned}$$

Solution: $(\hat{\lambda}_1, \hat{\lambda}_3) = (\frac{1}{2}, \frac{1}{2}) \implies \hat{\mathbf{x}} = (\frac{1}{2}, 0)^T, \hat{\pi} = \frac{1}{2}, \hat{q} = 0$

Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(0, 1)\mathbf{x}\} = 0 \implies$ Optimum for CP!

$$\begin{array}{l} \text{Fix variable values:} \quad x_1 = 0 \quad \text{or} \quad x_1 = 1 \\ \quad \quad \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\ \quad \quad \quad \quad \quad \quad \quad \lambda_3 = 0 \quad \quad \lambda_1 = 0 \end{array}$$

Branching, left (CP0): $\lambda_3 = 0$

$$\begin{array}{ll} \min & 0 \\ \text{s.t.} & 0 \geq 1 \\ & \lambda_1 = 1 \\ & \lambda_1 \geq 0 \end{array} \Rightarrow \left[\begin{array}{c} \text{infeasible} \\ \Downarrow \\ \text{add} \\ \text{column} \end{array} \right] \Rightarrow \begin{array}{ll} z_{CP0} \leq \min & 2\lambda_2 \\ \text{s.t.} & 2\lambda_2 \geq 1 \\ & \lambda_1 + \lambda_2 = 1 \\ & \lambda_1, \lambda_2 \geq 0 \end{array}$$

$$\begin{array}{ll} = \max & \pi + q \\ \text{s.t.} & q \leq 0 \\ & 2\pi + q \leq 2 \\ & \pi \geq 0 \end{array}$$

$$\begin{array}{l} \text{Solution: } (\hat{\lambda}_1, \hat{\lambda}_2) = \left(\frac{1}{2}, \frac{1}{2}\right) \\ \Rightarrow \hat{\mathbf{x}} = \left(0, \frac{1}{2}\right)^T \\ \hat{\pi} = 1, \quad \hat{q} = 0 \end{array}$$

Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(-1, 0)\mathbf{x} - 0\} = -1 < 0$

\Rightarrow New column! (λ_3 or λ_4 , but $\lambda_3 \equiv 0$) \Rightarrow Choose λ_4

$$\begin{aligned}
 z_{CP0} \leq \min \quad & 2\lambda_2 + 3\lambda_4 & = \quad & \max \quad \pi + q \\
 \text{s.t.} \quad & 2\lambda_2 + 4\lambda_4 \geq 1 & & \text{s.t.} \quad q \leq 0 \\
 & \lambda_1 + \lambda_2 + \lambda_4 = 1 & & 2\pi + q \leq 2 \\
 & \lambda_1, \lambda_2, \lambda_4 \geq 0 & & 4\pi + q \leq 3 \\
 & & & \pi \geq 0
 \end{aligned}$$

- ▶ Solution: $(\hat{\lambda}_1, \hat{\lambda}_3, \hat{\lambda}_4) = (\frac{3}{4}, 0, \frac{1}{4}) \implies \hat{\mathbf{x}} = (\frac{1}{4}, \frac{1}{4})^T, \hat{\pi} = \frac{3}{4}, \hat{q} = 0$
- ▶ Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(-\frac{1}{2}, \frac{1}{2})\mathbf{x}\} = -\frac{1}{2} \implies$
- ▶ Generate new column: λ_3 , but $\lambda_3 \equiv 0 \implies$ Optimum for CP0

Branching, right (CP1): $\lambda_1 = 0$

$$\begin{aligned} z_{CP1} \leq \min \quad & \lambda_3 \\ \text{s.t.} \quad & 2\lambda_3 \geq 1 \\ & \lambda_3 = 1 \\ & \lambda_3 \geq 0 \end{aligned} \quad = \quad \begin{aligned} \max \quad & \pi + q \\ \text{s.t.} \quad & 2\pi + q \leq 1 \\ & \pi \geq 0 \end{aligned}$$

- ▶ Solution: $\hat{\lambda}_3 = 1 \implies \hat{\mathbf{x}} = (1, 0)^T, \hat{\pi} = 0, \hat{q} = 1$
- ▶ Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(1, 2)\mathbf{x} - 1\} = -1 < 0 \implies$
- ▶ Generate new column: λ_1 , but $\lambda_1 \equiv 0 \implies$ Optimum for CP1

Branching, left, left: (CP00) $\lambda_2 = \lambda_4 = 0$

CP00: $\lambda_2 = \lambda_3 = \lambda_4 = 0 \implies$ infeasible

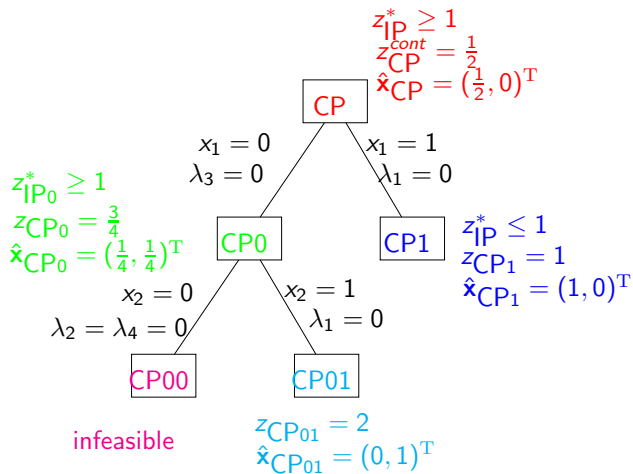
Branching, left, right: (CP01) $\lambda_1 = 0$

CP01: $\lambda_1 = \lambda_3 = 0$

$$\begin{aligned} z_{CP01} \leq \min \quad & 2\lambda_2 + 3\lambda_4 & = \quad & \max \quad \pi + q \\ \text{s.t.} \quad & 2\lambda_2 + 4\lambda_4 \geq 1 & \text{s.t.} \quad & 2\pi + q \leq 2 \\ & \lambda_2 + \lambda_4 = 1 & & 4\pi + q \leq 3 \\ & \lambda_2, \lambda_4 \geq 0 & & \pi \geq 0 \end{aligned}$$

- ▶ Solution: $(\hat{\lambda}_2, \hat{\lambda}_4) = (1, 0)^T \implies \hat{\mathbf{x}} = (0, 1)^T, \hat{\pi} = 0, \hat{q} = 2$
- ▶ Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(1, 2)\mathbf{x} - 2\} = -2 < 0$
 - \implies Generate new column: λ_1 , but $\lambda_1 \equiv 0$
 - \implies Generate new column: λ_3 , but $\lambda_3 \equiv 0$
 - \implies Optimum for CP01

Branch-and-price tree



Illustration

