The opportunistic maintenance problem

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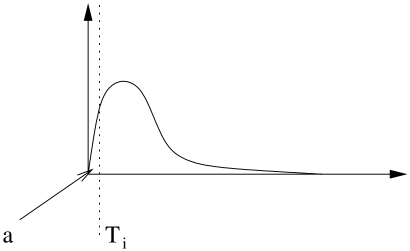
A motivating example

- Wind power turbine with 14 major components
- Crane is necessary for replacement of failed components
- Given failure of one component (opportunity), decide if other components should be replaced
- The decision is based on:
  - Components' life distributions (data)
  - Price of new component and maintenance occasion cost
  - Remaining life of the turbine

Deterministic component lives



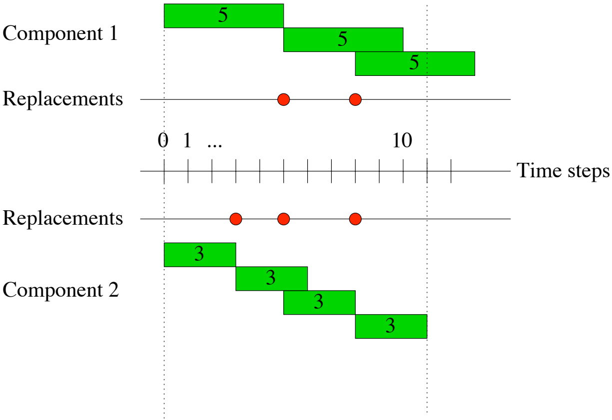
For each component $i \in \mathcal{N}$, choose the component life T_i such that

$$P(t \leq T_i) = a,$$

for a small a.

Deterministic component lives

cost: $2c_1 + 3c_2 + 3d$



Definition

Given lives T_i for every component i, costs c_{it}, d and timehorizon T, minimize the maintenance cost.

Solved by MILP model introduced by Dickman, Epstein and Wilamowsky (1991).

The variables

$$x_{it} = \begin{cases} 1 & \text{component } i \text{ is replaced at time } t, \\ 0 & \text{otherwise.} \end{cases}$$

$$z_t = \begin{cases} 1 & \text{maintenance performed at time } t, \\ 0 & \text{otherwise.} \end{cases}$$

The deterministic model

the objective function

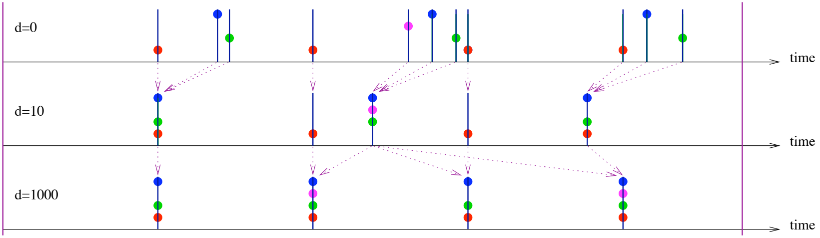
$$\text{minimize} \sum_{t \in \mathcal{T}} \left(\sum_{i \in \mathcal{N}} c_{it} x_{it} + d z_t \right),$$

the constraints

$$x_{it} \leq z_t \quad \color{red}{OR} \quad \sum_{i \in \mathcal{N}} x_{it} \leq N z_t \ \color{red}{?\ 2min!}$$

$$\begin{aligned} \sum_{t=l+1}^{l+T_i} x_{it} &\geq 1, \quad i \in \mathcal{N}, l \in \{0, \ldots, T - T_i\}, \\ x_{it} &\in \{0,1\}, \quad i \in \mathcal{N}, t \in \mathcal{T}, \\ z_t &\in \{0,1\}, \quad t \in \mathcal{T}. \end{aligned}$$

Small example



Properties

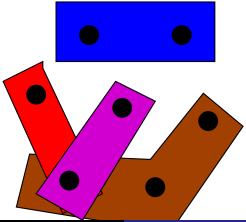
- Complexity: NP-hard.
- We can relax the integrality on x_{it}.
- If we fix z_t, we can use greedy.

NP, P and NPC.

- NP: decision problems verifiable in polynomial time.
- P: polynomially solvable problems (Ex. shortest path, LP, assignment problem...).
- NPC: If all problems in NP are polynomially reducible to problem A, A is in NPC.

Example (set covering decision problem)

Given: $A = \{1, \ldots, k\}$, $S_1, \ldots, S_\ell \subset A$.
Question: Is there cover of cardinality $<= N$?

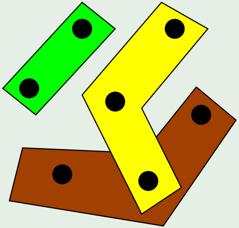


NP-hard

NP-hard: A is NPC and A is polynomially reducible to B
$\Rightarrow$ B is NP-hard

Example (set covering optimization problem)

- Given: $A = \{1, \ldots, k\}$, $S_1, \ldots, S_\ell \subset A$.
  Question: Which is the cover of smallest cardinality?



The replacement problem is NP-hard.

Theorem

Set covering is polynomially reducible to the replacement problem.

Proof.

- Consider the replacement problem with $n = k$, $T = \ell, T_i = \ell$, $d = 1$, $c_{it} = 0$ if $i \in S_t$ and $c_{it} = 2$ otherwise.
- Show that a solution to this RP yield an optimal solution to the SC, 10 min!

$\square$

Definition

A matrix A is TU if the determinant of each square submatrix is equal to -1,0 or 1.

Example

The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is TU.

The matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is not TU.

Theorem

- *If A is TU then (A, I) is TU.*
- *If A has consecutive ones property then it is TU.*
- *If A is TU then A^T is TU.*

Theorem

Consider $x^* = \arg\min\{c^T x | x \in \mathbb{Z}^n, Ax \le b\}$. *If A is TU, b is integral and the solution to the LP* $x^*_{LP} = \arg\min\{c^T x | x \in \mathbb{R}^n, Ax \le b\}$, *then* $x^* = x^*_{LP}$.

Proof.

- Constraint is equivalent to $Ax + Is = b$.
- Optimal basis B is a submatrix of (A, I).
- Optimal solution $(x_B, x_N) = (B^{-1}b, 0)$.
- Cramers rule $B^{-1} = \frac{B^*}{detB}$, B^* product of elements in B.
- B^{-1} is integral $\Rightarrow$ $B^{-1}b$ is integral.

$\square$

Relax integrality on x_{it}

Theorem

If z_t is fixed and integral, then there exists an optimal LP solution which yields integral values on x_{it}.

Proof.

- Prove this by considering the constraint matrix! 5min!

□

Greedy

Theorem

If the costs depend monotonically on time (i.e. $c_{it} \geq c_{it+1}$ or $c_{it} \leq c_{it+1}$ and the maintenance occasions are fixed (i.e. z_t are fixed), then a greedy algorithm yields and optimal maintenance schedule.

Proof.

Which greedy algorithm? 2 minutes! □

Discussion

Which algorithm could use the TU property and the greedy property?