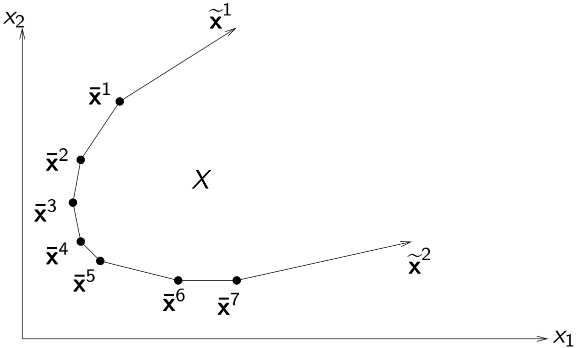
TMA521/MMA510
Optimization, project course
Lecture 9
Dantzig–Wolfe decomposition, column generation, and branch–and–price

Ann-Brith Strömberg

20 September 2011

Formulation of LP in a form suitable for column generation: Dantzig–Wolfe decomposition

- Let $X = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{Ax} = \mathbf{b}\}$ (or $\mathbf{Ax} \leq \mathbf{b}$) be a polyhedron with
- *extreme points* $\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$ and
- *extreme recession directions* $\widetilde{\mathbf{x}}^r$, $r \in \mathcal{R}$



Inner representation of the set X

$$\mathbf{x} \in X \iff \left(\begin{array}{rcl} \mathbf{x} & = & \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p + \sum_{r \in \mathcal{R}} \mu_r \tilde{\mathbf{x}}^r \\ & & \sum_{p \in \mathcal{P}} \lambda_p = 1 \\ & & \lambda_p \geq 0, \quad p \in \mathcal{P} \\ & & \mu_r \geq 0, \quad r \in \mathcal{R} \end{array} \right)$$

- $\mathbf{x} \in X$ is a *convex* combination of the extreme points plus a *conical* combination of the extreme directions
- Use this *inner representation* of the set X to reformulate an LP according to the *Dantzig-Wolfe decomposition principle*
- Solve by *column generation*

An LP and its corresponding complete master problem

$$
\begin{aligned}
\text{(LP1)} \quad z^* = \text{minimum} \ & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\
\text{subject to} \ \mathbf{D}\mathbf{x} &= \mathbf{d} \quad \longleftarrow \quad \text{(complicating constraints)} \\
\mathbf{A}\mathbf{x} &= \mathbf{b} \quad \longleftarrow \quad \text{(``simple'' constraints)} \\
\mathbf{x} &\geq \mathbf{0}
\end{aligned}
$$

- Let $X = \{ \mathbf{x} \geq 0 \mid \mathbf{A}\mathbf{x} = \mathbf{b} \}$
- Extreme points $\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$
- Extreme directions $\tilde{\mathbf{x}}^r$, $r \in \mathcal{R}$

$\Longrightarrow$

The complete master problem

$$
\begin{aligned}
\text{(LP2)}\ z^* = \min_{(\lambda,\mu)} \ & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{c}^{\mathrm{T}} \widetilde{\mathbf{x}}^r) \\
\text{s.t.} \ & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{D}\bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{D}\widetilde{\mathbf{x}}^r) = \mathbf{d} && | \ \boldsymbol{\pi} \\
& \sum_{p \in \mathcal{P}} \lambda_p = 1 && | \ q \\
& \lambda_p \geq 0, \ \ p \in \mathcal{P} \\
& \mu_r \geq 0, \ \ r \in \mathcal{R}
\end{aligned}
$$

- Number of constraints in (LP2) equals "the number of constraints in $\mathbf{Dx} = \mathbf{d}$" $+ 1$
- Number of columns very large ($= \#$ extreme points & directions of X)

The restricted master problem

- Assume that not all extreme points/directions are found: $\bar{\mathcal{P}} \subseteq \mathcal{P}$; $\bar{\mathcal{R}} \subseteq \mathcal{R}$

$$
\begin{aligned}
\text{(LP2-R)}\ z^* = \min_{(\lambda,\mu)} \ & \sum_{p \in \bar{\mathcal{P}}} \lambda_p (\mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}^p) + \sum_{r \in \bar{\mathcal{R}}} \mu_r (\mathbf{c}^{\mathrm{T}} \tilde{\mathbf{x}}^r) \\
\text{s.t.} \ & \sum_{p \in \bar{\mathcal{P}}} \lambda_p (\mathbf{D} \bar{\mathbf{x}}^p) + \sum_{r \in \bar{\mathcal{R}}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} && \mid \boldsymbol{\pi} \\
& \sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 && \mid q \\
& \lambda_p \geq 0, \ \ p \in \bar{\mathcal{P}} \\
& \mu_r \geq 0, \ \ r \in \bar{\mathcal{R}}
\end{aligned}
$$

- The number of constraints in (LP2) equals "the number of constraints in $\mathbf{D}\mathbf{x} = \mathbf{d}$" $+\ 1$
- The number of columns is "smaller"

The LP dual of the (restricted) master problem

- Assume that not all extreme points/directions are found: $\bar{\mathcal{P}} \subseteq \mathcal{P}$; $\bar{\mathcal{R}} \subseteq \mathcal{R}$
- The dual of (LP2-R) is given by

$$
\begin{aligned}
\text{(DLP2-R)}\ z^* \leq \max_{(\boldsymbol{\pi}, q)}\ & \mathbf{d}^{\mathrm{T}}\boldsymbol{\pi} + q \\
\text{s.t.}\ & (\mathbf{D}\bar{\mathbf{x}}^p)^{\mathrm{T}}\boldsymbol{\pi} + q \leq (\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p), \quad p \in \bar{\mathcal{P}} \quad | \lambda_p \\
& (\mathbf{D}\tilde{\mathbf{x}}^r)^{\mathrm{T}}\boldsymbol{\pi} \leq (\mathbf{c}^{\mathrm{T}}\tilde{\mathbf{x}}^r), \quad r \in \bar{\mathcal{R}} \quad | \mu_r
\end{aligned}
$$

with solution $(\bar{\boldsymbol{\pi}}, \bar{q})$

- Reduced cost for the variable λ_p, $p \in \mathcal{P} \setminus \bar{\mathcal{P}}$:
$$(\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p) - (\mathbf{D}\bar{\mathbf{x}}^p)^{\mathrm{T}}\bar{\boldsymbol{\pi}} - \bar{q} = (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\bar{\boldsymbol{\pi}})^{\mathrm{T}}\bar{\mathbf{x}}^p - \bar{q}$$
- Reduced cost for the variable μ_r, $r \in \mathcal{R} \setminus \bar{\mathcal{R}}$:
$$(\mathbf{c}^{\mathrm{T}}\tilde{\mathbf{x}}^r) - (\mathbf{D}\tilde{\mathbf{x}}^r)^{\mathrm{T}}\bar{\boldsymbol{\pi}} = (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\bar{\boldsymbol{\pi}})^{\mathrm{T}}\tilde{\mathbf{x}}^r$$

Column generation

- The smallest reduced cost is found by solving the column generation subproblem

$$\min_{\mathbf{x}\in X} (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\bar{\pi})^{\mathrm{T}}\mathbf{x} \quad \left(\text{alt:} \quad \min_{\mathbf{x}\in X} (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\bar{\pi})^{\mathrm{T}}\mathbf{x} - \bar{q}\right)$$

- Gives as solution an extreme point, $\bar{\mathbf{x}}^p$, or an extreme direction $\widetilde{\mathbf{x}}^r$ (Unbounded solutions can be detected within the simplex method! How?)

$\Longrightarrow$ a new column in (LP2) (if the reduced cost < 0):

- Either $\begin{pmatrix} \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p \\ \mathbf{D}\bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \mathbf{c}^{\mathrm{T}}\widetilde{\mathbf{x}}^r \\ \mathbf{D}\widetilde{\mathbf{x}}^r \\ 0 \end{pmatrix}$ enters the problem and improves the solution

A small example of DW decomposition and column generation

(IP)

$$\begin{array}{rrl} z^*_{\mathrm{IP}} & = \min & 2x_1 + 3x_2 + \ \ x_3 + 4x_4 \\ & \text{s.t.} & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 \qquad \mid \mathbf{Dx} = \mathbf{d} \\ & & \ \ x_1 + \ \ x_2 + \ \ x_3 + \ \ x_4 = 2 \\ & & \ \ x_1 \ , \ \ x_2 \ , \ \ x_3 \ , \ \ x_4 \in \{0,1\} \end{array}$$

- $X = \left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\} = \{\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^6\}$
- Optimal solution: $\mathbf{x}^*_{\mathrm{IP}} = (0,1,1,0)^{\mathrm{T}}$
- Optimal value: $z^*_{\mathrm{IP}} = 4$

LP-relaxation

(LP1)

$$
\begin{array}{rll}
z^* = \min & 2x_1 + 3x_2 + x_3 + 4x_4 & [\mathbf{c}^{\mathrm{T}}\mathbf{x}] \\
\text{s.t.} & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 & [\mathbf{Dx} = \mathbf{d}] \\
& x_1 + x_2 + x_3 + x_4 = 2 & [\mathbf{x} \in X] \\
0 \leq & x_1 \;,\; x_2 \;,\; x_3 \;,\; x_4 \leq 1 & [\mathbf{x} \in X]
\end{array}
$$

- $X = \operatorname{conv}\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix} \right\}$

$= \operatorname{conv}\{\bar{\mathbf{x}}^1, \ldots, \bar{\mathbf{x}}^6\}$

$$= \left\{ \mathbf{x} \in \mathbb{R}^4 \,\middle|\, \mathbf{x} = \sum_{p=1}^{6} \lambda_p \bar{\mathbf{x}}^p; \sum_{p=1}^{6} \lambda_p = 1; \lambda_p \geq 0, p = 1, \ldots, 6 \right\}$$

The complete master problem and the initial columns

(LP2)

$$
\begin{aligned}
z^* = \min\ & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 + 7\lambda_5 + 5\lambda_6 \\
\text{s.t.}\ & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 + 4\lambda_5 + 5\lambda_6 = 5 \\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 \\
& \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6 \geq 0
\end{aligned}
$$

- Initial columns: $\lambda_1, \lambda_2, \lambda_3$

(LP2-R)

$$
\begin{aligned}
z^* \leq \min\ & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 \\
\text{s.t.}\ & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 = 5 \\
& \lambda_1 + \lambda_2 + \lambda_3 = 1 \\
& \lambda_1,\ \lambda_2,\ \lambda_3 \geq 0
\end{aligned}
$$

(DLP2-R)

$$
\begin{aligned}
z^* \leq \max\ & 5\pi + q \\
\text{s.t.}\ & 5\pi + q \leq 5 \\
& 6\pi + q \leq 3 \\
& 5\pi + q \leq 6
\end{aligned}
$$

- Solution: $\bar{\boldsymbol{\lambda}} = (1, 0, 0)^{\mathrm{T}},$ $\bar{\pi} = -2, \quad \bar{q} = 15$

Reduced costs computation

$$\min_{\mathbf{x}\in X} \left\{ (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\bar{\pi})^{\mathrm{T}}\mathbf{x} - \bar{q} \right\} = \min_{p=1,\dots,6} \left\{ (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\bar{\pi})^{\mathrm{T}}\bar{\mathbf{x}}^p - \bar{q} \right\}$$
$$= \min_{p=1,\dots,6} \left\{ [(2,3,1,4) - (3,2,3,2)\cdot(-2)]\,\bar{\mathbf{x}}^p - 15 \right\}$$
$$= \min\{0,0,1,-1,0,0\} = -1 < 0$$

- New extreme point in (LP1): $\bar{\mathbf{x}}^4 = (0,1,1,0)^{\mathrm{T}}$
- New column in (LP2-R): $\begin{pmatrix} \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^4 \\ \mathbf{D}\bar{\mathbf{x}}^4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

New, extended problem

(LP2-R)

$$\begin{aligned} z^* \leq \min \;& 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 \\ \text{s.t. } \;& 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ & \lambda_1, \; \lambda_2, \; \lambda_3, \; \lambda_4 \geq 0 \end{aligned}$$

(DLP2-R)

$$\begin{aligned} z^* \leq \max \;& 5\pi + q \\ \text{s.t. } \;& 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \\ & 5\pi + q \leq 4 \end{aligned}$$

- Solution: $\bar{\boldsymbol{\lambda}} = (0,0,0,1)^{\mathrm{T}}$, $\bar{\pi} = -1$, $\bar{q} = 9$
- Reduced costs:
$$\min_{p=1,\dots,6} \left\{ (5,5,4,6)\,\bar{\mathbf{x}}^p - 9 \right\} = \min\{1,0,2,0,2,1\} = 0$$

Optimal solution to (LP2) and (LP1)

- $\boldsymbol{\lambda}^* = (0,0,0,1,0,0)^{\mathrm{T}}, \qquad \pi^* = -1, \quad q^* = 9$

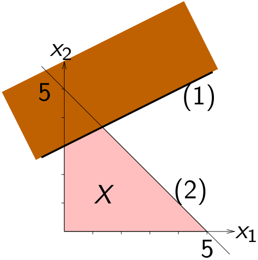
$\Longrightarrow$ $\mathbf{x}^* = \bar{\mathbf{x}}^4 = (0,1,1,0)^{\mathrm{T}} = \mathbf{x}^*_{\mathrm{IP}}, \qquad z^* = 4 = z^*_{\mathrm{IP}}$

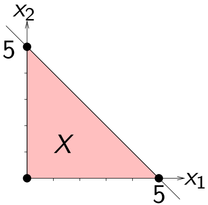
- A coincidence that the solution was integral!
- In general, the solution $\mathbf{x}^*$ to (LP1) may have fractional variable values
- **Solution to (IP)?**
- Need to find an integral solution (not certainly an optimal solution to (IP)) among the columns generated, i.e., solve

$$\min \left\{ (2,3,1,4)\mathbf{x} \,\middle|\, (3,2,3,2)\mathbf{x} = 5, \ \mathbf{x} \in \left\{ \bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \bar{\mathbf{x}}^3, \bar{\mathbf{x}}^4 \right\} \right\}$$

Another numerical example of Dantzig-Wolfe decomposition and column generation

$$\begin{array}{rrcrcrl} \min & x_1 & - & 3x_2 & & & (0) \\ \text{st} & -x_1 & + & 2x_2 & \leq & 6 & (1) \quad \longleftarrow \quad \text{(complicating)} \\ & x_1 & + & x_2 & \leq & 5 & (2) \\ & x_1 & , & x_2 & \geq & 0 & (3) \end{array}$$





$$X = \left\{ \mathbf{x} \in \mathbb{R}^2_+ \,\middle|\, x_1 + x_2 \leq 5 \right\} = \operatorname{conv}\left\{ (0,0)^{\mathrm{T}}, (0,5)^{\mathrm{T}}, (5,0)^{\mathrm{T}} \right\}$$

Complete DW master problem

$$\mathbf{x} \in X \iff \left\{ \begin{array}{r} \mathbf{x} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\lambda_3 \\ 5\lambda_2 \end{pmatrix} \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \right\}$$

$$\begin{array}{lrll} \min & -15\lambda_2 + 5\lambda_3 & & (0) \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 & \leq 6 & (1) \\ & \lambda_1 + \lambda_2 + \lambda_3 & = 1 & \\ & \lambda_1, \lambda_2, \lambda_3 & \geq 0 & \end{array}$$

- The first restricted master problem is then constructed from the points $(0,0)^{\mathrm{T}}$ and $(0,5)^{\mathrm{T}}$ (corresponds to λ_1 and λ_2)

Iteration 1

- $$\begin{array}{llrl} \min & - & 15\lambda_2 & (0) \\ \text{s.t.} & & 10\lambda_2 \le 6 & (1) \\ & & \lambda_1 + \lambda_2 = 1 & \\ & & \lambda_1, \lambda_2 \ge 0 & \end{array} \quad \Bigg| \quad \begin{array}{ll} \text{Solution:} & \bar{\boldsymbol{\lambda}} = (\frac{2}{5}, \frac{3}{5})^{\mathrm{T}} \\ \text{Dual solution:} & \bar{\pi} = -\frac{3}{2}, \bar{q} = 0 \end{array}$$

- Smallest reduced cost:

$$\min_{\mathbf{x} \in X} \left[(\mathbf{c}^{\mathrm{T}} - \bar{\pi}\mathbf{D})\mathbf{x} - \bar{q} \right] = \min_{\mathbf{x} \in X} \left(\left[(1, -3) - (-\tfrac{3}{2})(-1, 2) \right] \mathbf{x} - 0 \right)$$

$$= \min \left\{ -\tfrac{1}{2} x_1 \mid x_1 + x_2 \le 5; \mathbf{x} \ge \mathbf{0}^2 \right\} = -\tfrac{5}{2} < 0 \Longrightarrow \bar{\mathbf{x}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

- New column:

$$\left. \begin{array}{r} \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}} = (1, -3)(5, 0)^{\mathrm{T}} = 5 \\ \mathbf{D}\bar{\mathbf{x}} = (-1, 2)(5, 0)^{\mathrm{T}} = -5 \end{array} \right\} \Longrightarrow \begin{pmatrix} 5 \\ \hline -5 \\ 1 \end{pmatrix}$$

Iteration 2

$$\begin{array}{rr} \min & -15\lambda_2 + 5\lambda_3 \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \le 6 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \ge 0 \end{array} \quad \begin{array}{ll} \text{Solution:} & \bar{\boldsymbol{\lambda}} = (0, \frac{11}{15}, \frac{4}{15})^{\mathrm{T}} \\ \text{Dual solution:} & \bar{\pi} = -\frac{4}{3}, \bar{q} = -\frac{5}{3} \end{array}$$

- Smallest reduced cost:

$$\min_{\mathbf{x} \in X} \left[(\mathbf{c}^{\mathrm{T}} - \bar{\pi}\mathbf{D})\mathbf{x} - \bar{q} \right] = \min_{\mathbf{x} \in X} \left(\left[(1, -3) - (-\tfrac{4}{3})(-1, 2) \right] \mathbf{x} - (-\tfrac{5}{3}) \right)$$

$$= \min \left\{ -\tfrac{1}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{5}{3} \;\middle|\; x_1 + x_2 \le 5; \mathbf{x} \ge \mathbf{0}^2 \right\} = 0$$

- Optimal solution: $\boldsymbol{\lambda}^* = \left(0, \dfrac{11}{15}, \dfrac{4}{15}\right)^{\mathrm{T}}$

$$\Longrightarrow \mathbf{x}^* = (5\lambda_3, 5\lambda_2)^{\mathrm{T}} = (\tfrac{4}{3}, \tfrac{11}{3})^{\mathrm{T}}; \quad z^* = \tfrac{4}{3} - 3 \cdot \tfrac{11}{3} = -9\tfrac{2}{3}$$

DW decomposition applied to an LP with block-angular structure

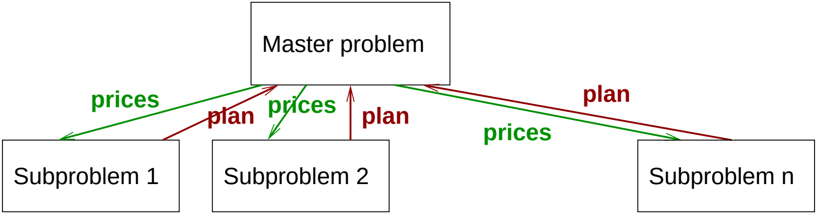
$$
\begin{array}{llll}
\max & \mathbf{c}_1^{\mathrm{T}}\mathbf{x}_1 + \mathbf{c}_2^{\mathrm{T}}\mathbf{x}_2 + \cdots + \mathbf{c}_n^{\mathrm{T}}\mathbf{x}_n & & \\
\text{s.t.} & \mathbf{D}_1\mathbf{x}_1 + \mathbf{D}_2\mathbf{x}_2 + \cdots + \mathbf{D}_n\mathbf{x}_n \leq \mathbf{d} & | & \text{Dual var: } \boldsymbol{\pi} \\
& \mathbf{A}_1\mathbf{x}_1 \leq \mathbf{b}_1 & | & \mathbf{x}_1 \in X_1 \\
& \mathbf{A}_2\mathbf{x}_2 \leq \mathbf{b}_2 & | & \mathbf{x}_2 \in X_2 \\
& \cdots \quad \cdot \; \cdot & & \\
& \mathbf{A}_n\mathbf{x}_n \leq \mathbf{b}_n & | & \mathbf{x}_n \in X_n \\
& \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \geq \mathbf{0} & &
\end{array}
$$

Cartesian product set:

$$X = X_1 \times X_2 \times \ldots \times X_n$$

DW decomposition as decentralized planning

- The main office (master problem) sets prizes ($\boldsymbol{\pi}$) for the common resources (complicating constraints)
- The departments (subproblems) suggest (production) plans (i.e., columns) ($\mathbf{D}_j \bar{\mathbf{x}}_j^p$) based on given prices
- The main office "mixes" the suggested plans (columns) optimally; sets new prices
- The procedure is repeated



Inner representations of the sets X_j, $j = 1, \ldots, n$

- Let $X_j = \{\mathbf{x}_j \geq \mathbf{0} \mid \mathbf{A}_j \mathbf{x}_j \leq \mathbf{b}_j\}$ and express $\mathbf{x}_j$ as

$$\mathbf{x}_j \in X_j \iff \left(\begin{array}{rcl} \mathbf{x}_j & = & \displaystyle\sum_{p \in \mathcal{P}_j} \lambda_{pj} \bar{\mathbf{x}}_j^p + \sum_{r \in \mathcal{R}_j} \mu_{rj} \tilde{\mathbf{x}}_j^r \\ & & \displaystyle\sum_{p \in \mathcal{P}_j} \lambda_{pj} = 1 \\ & & \lambda_{pj} \geq 0, \quad p \in \mathcal{P}_j \\ & & \mu_{rj} \geq 0, \quad r \in \mathcal{R}_j \end{array} \right) \quad j = 1, \ldots, n$$

$$\iff$$

- $X_j = \texttt{conv}\{\bar{\mathbf{x}}_j^p \mid p \in \mathcal{P}_j\} + \texttt{cone}\{\bar{\mathbf{x}}_j^r \mid r \in \mathcal{R}_j\}, \quad j = 1, \ldots, n$

The complete master problem

$$\max_{(\lambda,\mu)} \sum_{j=1}^{n} \left(\sum_{p \in \mathcal{P}_j} \lambda_{pj} (\mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}_j} \mu_{rj} (\mathbf{c}^{\mathrm{T}} \widetilde{\mathbf{x}}_j^r) \right)$$

$$\text{s.t.} \sum_{j=1}^{n} \left(\sum_{p \in \mathcal{P}_j} \lambda_{pj} (\mathbf{D} \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}_j} \mu_{rj} (\mathbf{D} \widetilde{\mathbf{x}}_j^r) \right) \leq \mathbf{d}$$

$$\sum_{p \in \mathcal{P}_j} \lambda_{pj} = 1, \qquad j = 1, \dots, n$$

$$\lambda_{pj} \geq 0, \ p \in \mathcal{P}_j, \ j = 1, \dots, n$$

$$\mu_{rj} \geq 0, \ r \in \mathcal{R}_j, \ j = 1, \dots, n$$

- The number of constraints in the master problem equals "the number of constraints in $\mathbf{Dx} = \mathbf{d}$" $+ \, n$
- The restricted master problem is formulated analogously as before (with $\bar{\mathcal{P}}_j \subseteq \mathcal{P}_j$ and $\bar{\mathcal{R}}_j \subseteq \mathcal{R}_j$, $j = 1, \dots, n$)

Use the LP dual solution to generate new columns

- The LP dual of the restricted master problem

$$
\begin{array}{lll}
\min\limits_{(\boldsymbol{\pi},\mathbf{q})} & \mathbf{d}^{\mathrm{T}}\boldsymbol{\pi} + \sum\limits_{j=1}^{n} q_j & \\
\text{s.t.} & (\mathbf{D}_j\bar{\mathbf{x}}_j^p)^{\mathrm{T}}\boldsymbol{\pi} + q_j \geq (\mathbf{c}_j^{\mathrm{T}}\bar{\mathbf{x}}_j^p), \quad p \in \bar{\mathcal{P}}_j,\ j = 1,\dots,n & \mid \lambda_{pj} \\
 & (\mathbf{D}_j\widetilde{\mathbf{x}}_j^r)^{\mathrm{T}}\boldsymbol{\pi} \geq (\mathbf{c}_j^{\mathrm{T}}\widetilde{\mathbf{x}}_j^r), \quad r \in \bar{\mathcal{R}}_j,\ j = 1,\dots,n & \mid \mu_{rj}
\end{array}
$$

with solution $\bar{\boldsymbol{\pi}}$, $\bar{q}_j$, $j = 1,\dots,n$

- Generate new columns (maximization $\Leftrightarrow$ reduced cost > 0):

For $j = 1,\dots,n$, solve $\boxed{\max\limits_{\mathbf{x}_j \in X_j} \left\{ (\mathbf{c}_j - \mathbf{D}_j^{\mathrm{T}}\bar{\boldsymbol{\pi}})^{\mathrm{T}}\mathbf{x}_j - \bar{q}_j \right\}}$

$\Rightarrow$ Solution $\bar{\mathbf{x}}_j^p$ or $\widetilde{\mathbf{x}}_j^r$

$\Rightarrow$ Column $\begin{pmatrix} \mathbf{c}_j^{\mathrm{T}}\bar{\mathbf{x}}_j^p \\ \hline \mathbf{D}_j^{\mathrm{T}}\bar{\mathbf{x}}_j^p \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \mathbf{c}_j^{\mathrm{T}}\widetilde{\mathbf{x}}_j^r \\ \hline \mathbf{D}_j^{\mathrm{T}}\widetilde{\mathbf{x}}_j^r \\ 0 \end{pmatrix}$ for each $j \in \{1,\dots,n\}$

Find feasible solutions (right-hand side allocation)

- Let $\bar{\lambda}_{pj}$, $p \in \mathcal{P}_j$, and $\bar{\mu}_{rj}$, $r \in \mathcal{R}_j$, $j = 1, \dots, n$, be a feasible and (almost) optimal solution to the restricted master problem
- It then holds that

$$\sum_{j=1}^{n} \mathbf{D}_j \underbrace{\left(\sum_{p \in \mathcal{P}} \bar{\lambda}_{pj} \bar{\mathbf{x}}_j^p + \sum_{r \in \mathcal{R}} \bar{\mu}_{rj} \tilde{\mathbf{x}}_j^r \right)}_{\in X_j} \leq \mathbf{d}$$

- Therefore, a good feasible $\mathbf{x}$-solution is given by the solution to the program

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}_j^{\mathrm{T}} \mathbf{x}_j \\ \text{subject to} \quad & \mathbf{D}_j \mathbf{x}_j \leq \sum_{p \in \mathcal{P}} \bar{\lambda}_{pj} (\mathbf{D}_j \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}} \bar{\mu}_{rj} (\mathbf{D}_j \tilde{\mathbf{x}}_j^r) \\ & \mathbf{x}_j \in X_j \end{aligned}$$

for $j = 1, \dots, n$

Upper bound on the optimal objective value for LP1

The complete master problem:

$$
\begin{aligned}
z^* = \min \ & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{c}^{\mathrm{T}} \widetilde{\mathbf{x}}^r) \\
\text{s.t.} \ & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{D} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{D} \widetilde{\mathbf{x}}^r) = \mathbf{d} \qquad | \ \boldsymbol{\pi} \\
& \sum_{p \in \mathcal{P}} \lambda_p = 1 \qquad | \ q \\
& \lambda_p, \mu_r \geq 0, \ p \in \mathcal{P}, r \in \mathcal{R}
\end{aligned}
$$

The dual of the restricted master problem:

$$
\begin{aligned}
z^* \leq \bar{z} := \mathbf{d}^{\mathrm{T}} \bar{\boldsymbol{\pi}} + \bar{q} = \max_{(\boldsymbol{\pi}, q)} \ & \mathbf{d}^{\mathrm{T}} \boldsymbol{\pi} + q \\
\text{s.t.} \ & (\mathbf{D} \bar{\mathbf{x}}^p)^{\mathrm{T}} \boldsymbol{\pi} + q \leq \mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}^p, \quad p \in \bar{\mathcal{P}} \\
& (\mathbf{D} \widetilde{\mathbf{x}}^r)^{\mathrm{T}} \boldsymbol{\pi} \leq \mathbf{c}^{\mathrm{T}} \widetilde{\mathbf{x}}^r, \quad r \in \bar{\mathcal{R}}
\end{aligned}
$$

Lower bound on the optimal objective value for LP1

- Let λ_p^*, $p \in \mathcal{P}$, and μ_r^*, $r \in \mathcal{R}$, be optimal in the complete master problem
- Let $(\bar{\pi}, \bar{q})$ be an optimal dual solution for the restricted master problem, with columns corresponding to $\bar{\mathcal{P}}$ and $\bar{\mathcal{R}}$
- Multiply the right-hand side elements of the primal (i.e., $\mathbf{d}$ and 1) by $\bar{\pi}$ and $\bar{q}$, respectively

$\Longrightarrow$

$$
\begin{aligned}
0 \geq\ & z^* - \bar{z} = z^* - \mathbf{d}^{\mathrm{T}}\bar{\pi} - 1 \cdot \bar{q} \\
=\ & \sum_{p \in \mathcal{P}} \lambda_p^* \left[\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p - (\mathbf{D}\bar{\mathbf{x}}^p)^{\mathrm{T}}\bar{\pi} - \bar{q} \right] + \sum_{r \in \mathcal{R}} \mu_r^* \left[\mathbf{c}^{\mathrm{T}}\widetilde{\mathbf{x}}^r - (\mathbf{D}\widetilde{\mathbf{x}}^r)^{\mathrm{T}}\bar{\pi} \right] \\
\geq\ & \min_{p \in \mathcal{P}} \left[\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p - (\mathbf{D}\bar{\mathbf{x}}^p)^{\mathrm{T}}\bar{\pi} - \bar{q} \right] + \sum_{r \in \mathcal{R}} \mu_r^* \min_{s \in \mathcal{R}} \left[\mathbf{c}^{\mathrm{T}}\widetilde{\mathbf{x}}^s - (\mathbf{D}\widetilde{\mathbf{x}}^s)^{\mathrm{T}}\bar{\pi} \right]
\end{aligned}
$$

Lower bound on the optimal objective value for LP1

- If the subproblem has an unbounded solution no optimistic estimate can be computed in this iteration
- Otherwise it holds that
$$\min_{s \in \mathcal{R}} \left[\mathbf{c}^{\mathrm{T}} \widetilde{\mathbf{x}}^s - (\mathbf{D}\widetilde{\mathbf{x}}^s)^{\mathrm{T}} \bar{\pi} \right] \geq 0$$

$\Longrightarrow$

$$\begin{aligned} \bar{z} \geq z^* \geq \bar{z} + \min_{p \in \mathcal{P}} & \left[(\mathbf{c} - \mathbf{D}^{\mathrm{T}} \bar{\pi})^{\mathrm{T}} \bar{\mathbf{x}}^p - \bar{q} \right] \\ &= \bar{z} + \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^{\mathrm{T}} \bar{\pi})^{\mathrm{T}} \mathbf{x} - \bar{q} \\ &=: \underline{z} \end{aligned}$$

Convergence of the column generation algorithm

- The number of columns generated is finite, since X is polyhedral
- When no more columns are generated, the solution to the last restricted master problem will also solve the original LP
- For each new column that is added to the master problem, its optimal objective value will decrease (or be kept constant)

$\Rightarrow$ The pessimistic estimate $\bar{z}_k$ converges monotonically to z^*

- The optimistic estimate $\underline{z}_k$ also converges, but not monotonically
- If at iteration k an optimal solution to the complete master problem is received, then $\underline{z}_k = \bar{z}_k$ holds
- Stopping criterion: $\bar{z}_k - \underline{z}_k^* \leq \varepsilon$, where $\underline{z}_k^* = \max_{s=1,\ldots,k} \underline{z}_s$ and $\varepsilon > 0$

Branch–and–price for linear 0/1 problems

(IP)

$$
\begin{aligned}
z_{\mathrm{IP}}^* = \min\ & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\
\text{s.t.}\ & \mathbf{D}\mathbf{x} = \mathbf{d} \\
& \mathbf{x} \in X = \{\mathbf{x} \in \mathbb{B}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\} = \{\bar{\mathbf{x}}^p \mid p \in \mathcal{P}\}
\end{aligned}
$$

- Inner representation (and convexification):

$$
\operatorname{conv} X = \left\{ \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p \,\middle|\, \sum_{p \in \mathcal{P}} \lambda_p = 1;\ \lambda_p \geq 0,\ p \in \mathcal{P} \right\}
$$

- Let $c_p = \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p$ and $\mathbf{d}_p = \mathbf{D}\bar{\mathbf{x}}^p$, $p \in \mathcal{P}$.

Stronger formulation—Master problem

(CP)

$$\begin{aligned} z_{\mathrm{IP}}^* = z_{\mathrm{CP}}^* = \min \ & \sum_{p \in \mathcal{P}} c_p \lambda_p \\ \text{s.t.} \ & \sum_{p \in \mathcal{P}} \mathbf{d}_p \lambda_p = \mathbf{d} \\ & \sum_{p \in \mathcal{P}} \lambda_p = 1 \\ & \lambda_p \in \{0,1\}, \qquad p \in \mathcal{P} \end{aligned}$$

- A continuous relaxation ((CP^{cont}), to $\lambda_p \geq 0$) of (CP) gives the same lower bound as the Lagrangian dual with respect to the constraints $\mathbf{Dx} = \mathbf{d}$ $\quad (z_{LP}^* \leq z_{CP}^{cont} \leq z_{CP}^*)$
- The continuous relaxation (LP) of (IP) is never better than any Lagrangian dual bound.

Restricted master problem

- Let $\bar{\mathcal{P}} \subseteq \mathcal{P}$

$$(\overline{\text{CP}}) \quad z^*_{\text{CP}} \geq z^{cont}_{\text{CP}} \leq \bar{z}_{\text{CP}} = \min \sum_{p \in \bar{\mathcal{P}}} c_p \lambda_p$$

$$\text{s.t.} \sum_{p \in \bar{\mathcal{P}}} \mathbf{d}_p \lambda_p = \mathbf{d}$$

$$\sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 \qquad (*)$$

$$\lambda_p \geq 0, \qquad p \in \bar{\mathcal{P}}$$

- Generate columns $\begin{pmatrix} c_p \\ \mathbf{d}_p \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{c}^{\mathrm{T}} \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ until an (almost) optimal solution to (CP^{cont}), $\widehat{\lambda}_p$ ($p \in \bar{\mathcal{P}}$), is found
- The corresponding $\mathbf{x}$-solution: $\widehat{\mathbf{x}} = \sum_{p \in \bar{\mathcal{P}}} \widehat{\lambda}_p \bar{\mathbf{x}}^p$

Branching over variable x_j with $0 < \hat{x}_j < 1$

$$x_j = 0 \qquad \text{or} \qquad x_j = 1$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 0 \qquad x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 1$$

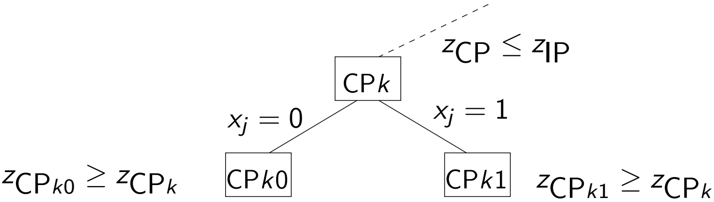
$$\Downarrow \qquad\qquad \Downarrow$$

delete col's $\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 1} \lambda_p = 0$ $\qquad$ $\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 1} \lambda_p = 1$ replaces $(*)$

$$\Updownarrow \qquad\qquad \Updownarrow$$

replaces $(*)$ $\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 0} \lambda_p = 1$ $\qquad$ $\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 0} \lambda_p = 0$ delete col's

Generate columns in B&B nodes



- In each node (CP, CP0, CP1, ...): Generate columns until (almost) optimal (all reduced costs ≥ 0) or verified infeasible
- If $\mathbf{x}^*_{CPk\ell\ldots}$ feasible $\Longrightarrow z^*_{CPk\ell\ldots} \geq z^*_{IP} \Longrightarrow$ Cut off the branch $(k, \ell, \ldots)$

$\Longrightarrow$ Cut branches $(r, s, \ldots)$ with $z^*_{CPrs\ldots} \geq z^*_{CPk\ell\ldots}$

The column generation subproblem, reduced costs

- $\min_{\mathbf{x}\in X^k} (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\widehat{\boldsymbol{\pi}}^k)^{\mathrm{T}}\mathbf{x} - \widehat{q}^k =: (\mathbf{c} - \mathbf{D}^{\mathrm{T}}\widehat{\boldsymbol{\pi}}^k)^{\mathrm{T}}\bar{\mathbf{x}}^p - \widehat{q}^k =: \bar{c}(\bar{\mathbf{x}}^p)$
- $(\widehat{\boldsymbol{\pi}}^k, \widehat{q}^k)$ is a dual solution to the RMP and $X^k = X \cap \{\mathbf{x} \mid x_j = k\}$, $k \in \{0,1\}$ (etc. down the tree)
- If $\bar{c}(\bar{\mathbf{x}}^p) < 0$ then $\begin{pmatrix} \mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}}^p \\ \mathbf{D}\bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$ is a new column in [CPk]
- Minimization? $\bar{\mathbf{x}}^r$ is good enough if $\bar{c}(\bar{\mathbf{x}}^r) < 0$
- If $\bar{c}(\bar{\mathbf{x}}^p) \geq 0$ then no more columns are needed to solve [CPk] to optimality.
- The same columns may be generated in different nodes $\Longrightarrow$ create “column pool” to check w.r.t. reduced costs $\bar{c}$

An instance solved by Branch–and–price

$$z^*_{IP} = \min \; x_1 + 2x_2 \quad \text{s.t.} \; 2x_1 + 2x_2 \geq 1, \; x_1, x_2 \in \{0,1\} \quad = z^*_{CP} \geq z^{cont}_{CP} = z^*_{LP} = \min \; x_1 + 2x_2 \quad \text{s.t.} \; 2x_1 + 2x_2 \geq 1, \; 0 \leq x_1, x_2 \leq 1$$

$$\text{conv} X = \text{conv}\left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\} = \left\{ \begin{pmatrix}\lambda_3+\lambda_4\\ \lambda_2+\lambda_4\end{pmatrix} \,\middle|\, \sum_{p=1}^{4} \lambda_p = 1; \; \begin{matrix}\lambda_p \geq 0 \\ \forall p\end{matrix} \right\}$$

[CP]

$$\begin{aligned} z^{cont}_{CP} = \min \quad & 2\lambda_2 + \lambda_3 + 3\lambda_4 \\ \text{s.t.} \quad & 2\lambda_2 + 2\lambda_3 + 4\lambda_4 \geq 1 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

Initial columns: λ_1 and λ_3

Choose e.g., $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, that is, the variables λ_1 and λ_3

$$
\begin{array}{rrcl}
z_{CP}^{cont} \le \min & \lambda_3 & = & \max \ \pi + q \\
\text{s.t.} & 2\lambda_3 \ge 1 & & \text{s.t.} \quad q \le 0 \\
& \lambda_1 + \lambda_3 = 1 & & 2\pi + q \le 1 \\
& \lambda_1, \lambda_3 \ge 0 & & \pi \ge 0
\end{array}
$$

Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_3) = (\frac{1}{2}, \frac{1}{2}) \Longrightarrow \widehat{\mathbf{x}} = (\frac{1}{2}, 0)^{\mathrm{T}}$, $\widehat{\pi} = \frac{1}{2}$, $\widehat{q} = 0$
Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(0,1)\mathbf{x}\} = 0 \Longrightarrow$ Optimum for CP!

Fix variable values: $x_1 = 0$ or $x_1 = 1$

$$
\begin{array}{ccc}
x_1 = 0 & & x_1 = 1 \\
\Downarrow & & \Downarrow \\
\lambda_3 = 0 & & \lambda_1 = 0
\end{array}
$$

Branching, left (CP0): $\lambda_3 = 0$

$$\begin{array}{rr} \min & 0 \\ \text{s.t.} & 0 \geq 1 \\ & \lambda_1 = 1 \\ & \lambda_1 \geq 0 \end{array} \implies \begin{bmatrix} \text{infeasible} \\ \Downarrow \\ \text{add} \\ \text{column} \end{bmatrix} \implies z_{CP0} \leq \begin{array}{rr} \min & 2\lambda_2 \\ \text{s.t.} & 2\lambda_2 \geq 1 \\ & \lambda_1 + \lambda_2 = 1 \\ & \lambda_1, \lambda_2 \geq 0 \end{array}$$

$$= \begin{array}{rr} \max & \pi + q \\ \text{s.t.} & q \leq 0 \\ & 2\pi + q \leq 2 \\ & \pi \geq 0 \end{array}$$

Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_2) = (\frac{1}{2}, \frac{1}{2})$
$\implies \widehat{\mathbf{x}} = (0, \frac{1}{2})^{\mathrm{T}}$
$\widehat{\pi} = 1, \quad \widehat{q} = 0$

Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(-1, 0)\mathbf{x} - 0\} = -1 < 0$

$\implies$ New column! (λ_3 or λ_4, but $\lambda_3 \equiv 0$) $\implies$ Choose λ_4

$$z_{CP0} \le \min \quad 2\lambda_2 + 3\lambda_4 \qquad = \quad \max \quad \pi + q$$

$$\text{s.t.} \quad 2\lambda_2 + 4\lambda_4 \ge 1 \qquad \text{s.t.} \quad q \le 0$$

$$\lambda_1 + \lambda_2 + \lambda_4 = 1 \qquad 2\pi + q \le 2$$

$$\lambda_1, \lambda_2, \lambda_4 \ge 0 \qquad 4\pi + q \le 3$$

$$\pi \ge 0$$

- Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_3, \widehat{\lambda}_4) = (\frac{3}{4}, 0, \frac{1}{4}) \Longrightarrow \widehat{\mathbf{x}} = (\frac{1}{4}, \frac{1}{4})^{\mathrm{T}}$, $\widehat{\pi} = \frac{3}{4}$, $\widehat{q} = 0$
- Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(-\frac{1}{2}, \frac{1}{2})\mathbf{x}\} = -\frac{1}{2} \Longrightarrow$
- Generate new column: λ_3, but $\lambda_3 \equiv 0 \Longrightarrow$ Optimum for CP0

Branching, right (CP1): $\lambda_1 = 0$

$$z_{CP1} \leq \min \; \lambda_3 \quad \text{s.t.} \; 2\lambda_3 \geq 1, \; \lambda_3 = 1, \; \lambda_3 \geq 0 \quad = \quad \max \; \pi + q \quad \text{s.t.} \; 2\pi + q \leq 1, \; \pi \geq 0$$

- Solution: $\widehat{\lambda}_3 = 1 \implies \widehat{\mathbf{x}} = (1,0)^{\mathrm{T}}$, $\widehat{\pi} = 0$, $\widehat{q} = 1$
- Reduced costs: $\min_{\mathbf{x}\in[0,1]^2} \{(1,2)\mathbf{x} - 1\} = -1 < 0 \implies$
- Generate new column: λ_1, but $\lambda_1 \equiv 0 \implies$ Optimum for CP1

Branching, left, left: (CP00) $\lambda_2 = \lambda_4 = 0$

CP00: $\lambda_2 = \lambda_3 = \lambda_4 = 0 \implies$ infeasible

Branching, left, right: (CP01) $\lambda_1 = 0$

CP01: $\lambda_1 = \lambda_3 = 0$

$$
z_{CP01} \leq \min \; 2\lambda_2 + 3\lambda_4 \quad \text{s.t.} \; 2\lambda_2 + 4\lambda_4 \geq 1, \; \lambda_2 + \lambda_4 = 1, \; \lambda_2, \lambda_4 \geq 0 \quad = \quad \max \; \pi + q \quad \text{s.t.} \; 2\pi + q \leq 2, \; 4\pi + q \leq 3, \; \pi \geq 0
$$

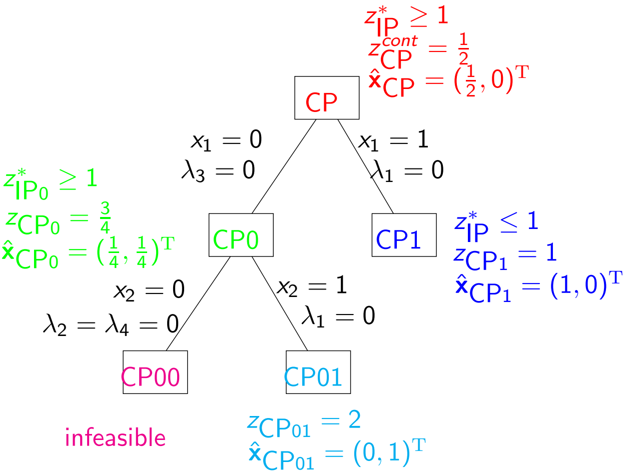
- Solution: $(\widehat{\lambda}_2, \widehat{\lambda}_4) = (1, 0)^{\mathrm{T}} \Longrightarrow \hat{\mathbf{x}} = (0, 1)^{\mathrm{T}}$, $\widehat{\pi} = 0$, $\widehat{q} = 2$
- Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(1, 2)\mathbf{x} - 2\} = -2 < 0$

  $\Longrightarrow$ Generate new column: λ_1, but $\lambda_1 \equiv 0$

  $\Longrightarrow$ Generate new column: λ_3, but $\lambda_3 \equiv 0$

  $\Longrightarrow$ Optimum for CP01

Branch–and–price tree



Illustration