TMA521/MMA511 Large Scale Optimization Lecture 1 Introduction: simple/difficult problems, matroid problems

Ann-Brith Strömberg
Professor of Mathematical Optimization

2018–01–16

TMA521/MMA511 Optimization, project course

- Examiner & lecturer
  Ann-Brith Strömberg (room L2087, anstr@chalmers.se)
- Schedule
  - 12 lectures
  - 2 workshops (mandatory presence)
  - 2–4 student presentations (mandatory presence)
- Course homepage
  `www.math.chalmers.se/Math/Grundutb/CTH/tma521/1718/`
- Two projects
  - Lagrangian relaxation for a VLSI design problem (Matlab)
  - Column generation applied to a real production scheduling problem (AMPL/Cplex, Matlab)

Requirements for passing the course and examination

1. You must take active part in the two scheduled workshops, and in the two course projects.
2. The findings of your projects are to be presented in the form of written reports as well as orally during seminars
3. Each group must also act as opponents/discussants on another group's projects, and each student must take active part in all of these activities
4. Define project groups in PingPong
5. All handing in of programs and reports is made in PingPong: "TMA521/MMA511 Large-Scale optimization Spring 18"
6. Project groups consist of two (or, if necessary, one) persons
7. To reach a mark higher than "pass" (i.e., 3 or G) you can take an oral exam, based on the lecture material of the course and on the projects. Most of the topics to discuss during such an exam are gathered in the study material (to appear on the homepage)

Course literature

- *Optimization Theory for Large Systems*, by L.S. Lasdon, Dover Publications 2002, ISBN: 9780486419992.
- Material on Lagrangean duality is also found in the book *An Introduction to Continuous Optimization* by N. Andréasson, A. Evgrafov, M. Patriksson, E. Gustavsson, Z. Nedělková, K.C. Sou, and M. Önnheim. Studentlitteratur (2016). Available at Cremona.
- *Articles and book chapters* (hand-outs/links on the course homepage)
- *Lecture notes* (published on the homepage)

Aim

- Purpose of the course: provide an overview of the most important principles for the efficient solution of *practical large-scale optimization problems*, from modelling to method implementation.

- Starting with a series of lectures, the course work is then gradually concentrated on project work, in which the knowledge gained is applied to efficiently solve some relevant large-scale optimization problems

Learning outcome

- After completion of this course, the student should
  - be able to independently analyze and suggest modelling and method principles for a variety of practical large-scale optimization problems, and
  - have suffient knowledge to utilize these principles successfully in practice through the use of optimization software tools.

Content

- Most large scale optimization problems have inherent structures that can be exploited to solve them efficiently
- The course deals with some principles through which large scale optimization problems can be attacked
- Techniques: decomposition–coordination. Based on convexity theory and duality.
- Three practical moments:
  - An exercise on modelling and solution of a design problem (Wednesday, January 24)
  - Two project assignments, where large scale optimization problems are solved using duality theory and techniques presented during the lectures

Content, cont'd.

- Problems: Complexity, unimodularity, convexity, minimal spanning trees, knapsack problems, location problems, generalized assignment, travelling salesperson problems, network design, set covering problems
- Principles: Decomposition & coordination, restriction, projection, variable fixing, neighbourhood, Lagrange relaxation, coordinating master problem
- Methods: Cutting planes, Lagrangian heuristics, column generation, Dantzig-Wolfe decomposition, Benders decomposition, local search, modern tree search methods

Topics: Turn difficult problems into sequences of simpler ones using decomposition and coordination

Prerequisites

- Linear Programming (LP), [Mixed] Integer Linear programming ([M]ILP), NonLinear Programming (NLP),

Decomposition methods covered

- Lagrangian relaxation (for MILP, NLP)
- Dantzig–Wolfe decomposition (for LP)
- Column generation (for LP, MILP, NLP)
- Benders decomposition (for MILP, NLP)
- Heuristics (for ILP)
- Branch & Bound (for MILP, non-convex NLP)
- Greedy algorithms (for ILP, NLP)
- Subgradient optimization (for convex NLP, Lagrangian duals)

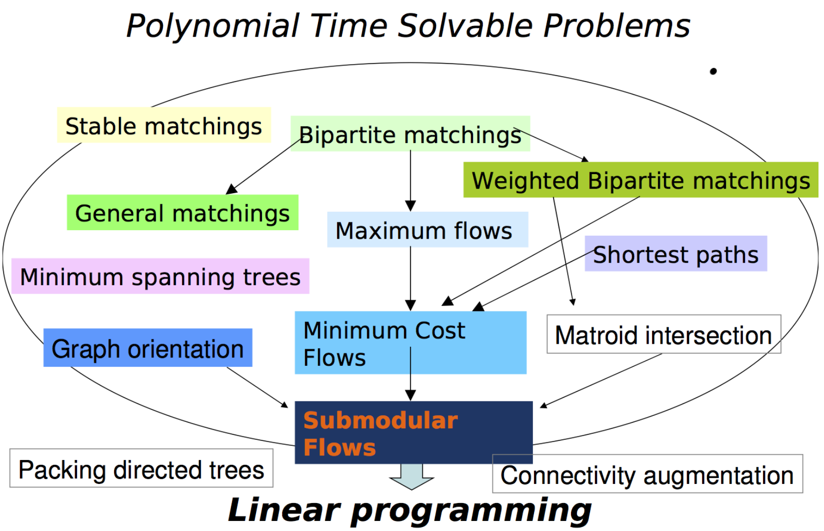
Important properties of simple problems

- What we here call *simple problems* can be solved in polynomial time with respect to the problem size (i.e., numbers of variables, constraints, ...)
- For *simple problems*, there exist *polynomial algorithms* preferably with a small largest exponent
- E.g., sorting an array of n elements can be done in time proportional to at most (i.e., worst case performance)
    - n^2 operations (bubble sort)
    - $n \log n$ operations (heapsort, quicksort)

Examples of simple problems

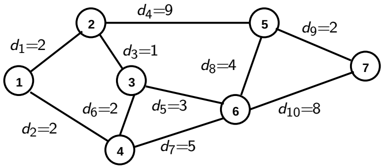
- Network flow problems (see Wolsey):
  - Shortest paths
  - Maximum flows
  - Minimum cost (single-commodity) network flows
  - The transportation problem
  - The assignment problem
  - Maximum cardinality matching
- Problems over simple matroids (see Lawler)
- Linear programming (see Andréasson et al.)

Polynomial time solvable problems



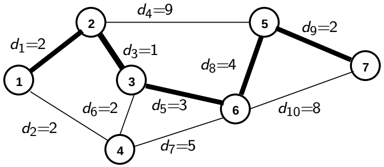
Example: Shortest path

Find the shortest path from node 1 to node 7



d_i = length of edge i

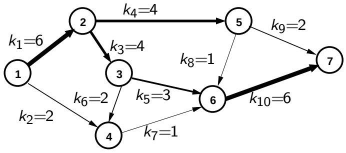
Shortest path from node 1 to node 7



Total length: 12

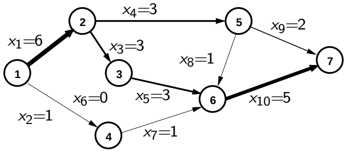
Example: Maximum flow

Find the maximum flow from node 1 to node 7



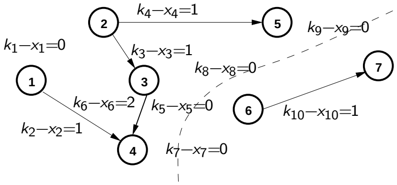
k_i = flow capacity of arc i (min kapac: 0)

Maximum flow from node 1 to node 7



x_i = optimal flow through arc i (≥ 0)

Minimum cut separating nodes 1 and 7



$k_i - x_i$ = residual flow capacity on arc i

Matroids and the greedy algorithm (Lawler)

- Greedy algorithm
  - Create a “complete solution” by iteratively choosing the best alternative
  - Never regret a previous choice
- Which problems can be solved using such a simple method?
- Problems whose feasible sets can be described by matroids

Matroids and independent sets

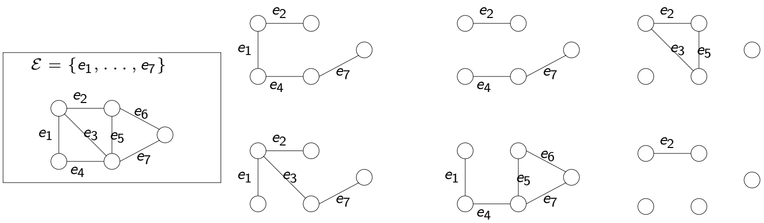
- Given a finite set $\mathcal{E}$ and a family $\mathcal{F}$ of subsets of $\mathcal{E}$: If $\mathcal{I} \in \mathcal{F}$ and $\mathcal{I}' \subseteq \mathcal{I}$ imply $\mathcal{I}' \in \mathcal{F}$, then the elements of $\mathcal{F}$ are called independent
- A matroid $M = (\mathcal{E}, \mathcal{F})$ is a structure in which $\mathcal{E}$ is a finite set of elements and $\mathcal{F}$ is a family of subsets of $\mathcal{E}$, such that
  1. $\emptyset \in \mathcal{F}$ and all proper subsets of a set $\mathcal{I}$ in $\mathcal{F}$ are in $\mathcal{F}$
  2. If $\mathcal{I}_p$ and $\mathcal{I}_{p+1}$ are sets in $\mathcal{F}$ with $|\mathcal{I}_p| = p$ and $|\mathcal{I}_{p+1}| = p + 1$, then $\exists$ an element $e \in \mathcal{I}_{p+1} \setminus \mathcal{I}_p$ such that $\mathcal{I}_p \cup \{e\} \in \mathcal{F}$
- Let $M = (\mathcal{E}, \mathcal{F})$ be a matroid and $\mathcal{A} \subseteq \mathcal{E}$: If $\mathcal{I}$ and $\mathcal{I}'$ are maximal independent subsets of $\mathcal{A}$, then $|\mathcal{I}| = |\mathcal{I}'|$

Example I: Matric matroids

- $\mathcal{E}$ = a set of column vectors in $\mathbb{R}^n$
- $\mathcal{F}$ = the set of linearly independent subsets of vectors in $\mathcal{E}$.
- Let $n = 3$ and $\mathcal{E} = [e_1, \dots, e_5] = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}$
- We have:
  - $\{e_1, e_2, e_3\} \in \mathcal{F}$ and $\{e_2, e_3\} \in \mathcal{F}$ but
  - $\{e_1, e_2, e_3, e_5\} \notin \mathcal{F}$ and $\{e_1, e_4, e_5\} \notin \mathcal{F}$

Example II: Graphic matroids

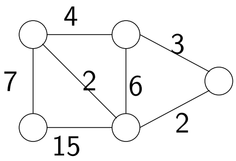
- $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ = the set of edges in an undirected graph
- $\mathcal{F}$ = the set of all cycle-free subsets of edges in $\mathcal{E}$

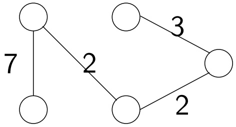


- $\{e_1, e_2, e_4, e_7\} \in \mathcal{F}$, $\{e_2, e_4, e_7\} \in \mathcal{F}$, $\{e_2, e_3, e_5\} \notin \mathcal{F}$, $\{e_1, e_2, e_3, e_7\} \in \mathcal{F}$, $\{e_1, e_4, e_5, e_6, e_7\} \notin \mathcal{F}$, $\{e_2\} \in \mathcal{F}$

Matroids and the greedy algorithm applied to Example II

- Let $w(e)$ be the cost of element $e \in \mathcal{E}$.
  Problem: Find the element $\mathcal{I} \in \mathcal{F}$ of maximal cardinality such that the total cost is at minimum/maximum
- Example II, continued: $w(\mathcal{E}) = (7, 4, 2, 15, 6, 3, 2)$





An element $\mathcal{I} \in \mathcal{F}$ of *maximal cardinality* with *minimum total cost*

The greedy algorithm for minimization problems

1. $\mathcal{A} = \emptyset$.
2. Sort the elements of $\mathcal{E}$ in increasing order with respect to $w(e)$.
3. Take the first element $e \in \mathcal{E}$ in the list. If $\mathcal{A} \cup \{e\}$ is still independent $\Longrightarrow$ let $\mathcal{A} := \mathcal{A} \cup \{e\}$.
4. Repeat from step 3. with the next element—until either the list is empty, or $\mathcal{A}$ possesses the maximal cardinality.

Which are the special cases of this algorithm for Examples I and II?

Example I: Linearly independent vectors—matric matroids

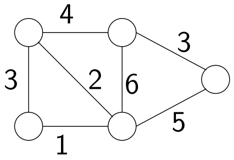
- Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 1 & 5 & 0 & 2 \end{pmatrix},$$
$$\boldsymbol{w}^{\mathrm{T}} = \begin{pmatrix} 10 & 9 & 8 & 4 & 1 \end{pmatrix}.$$

- Choose the maximal independent set with the maximum weight
- Can this technique solve linear programming problems?

Example II: minimum spanning trees (MST)—graphic matroids

- The maximal cycle-free set of links in an undirected graph is a spanning tree
- In a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, a spanning tree has $|\mathcal{N}| - 1$ links
- Classic greedy algorithm—Kruskal's algorithm has complexity $O(|\mathcal{E}| \cdot \log(|\mathcal{E}|))$. The main complexity lies in the sorting itself
- Prim's algorithm builds the spanning tree through graph search techniques, from node to node; complexity $O(|\mathcal{N}|^2)$.



Example III: continuous knapsack problem (in fact not a matroid problem)

- Continuous relaxation of the 0/1-knapsack problem (BKP). Assume: $c_j \geq 0$, $a_j > 0$, $j = 1, \ldots, n$; $b \geq 0$

$$
\begin{aligned}
\text{maximize } f(\boldsymbol{x}) :=& \sum_{j=1}^{n} c_j x_j, \\
\text{subject to } & \sum_{j=1}^{n} a_j x_j \leq b, \\
& 0 \leq x_j \leq 1, \qquad j = 1, \ldots, n.
\end{aligned}
$$

- Greedy algorithm:
  1. Sort c_j / a_j in descending order
  2. Set the variables $\bar{x}_j := 1$ until the knapsack is full
  3. One variable may become fractional and the rest zero

Discussion

- Discuss with each other for a few minutes:

- Use linear programming duality to show that the greedy algorithm correctly solves the continuous knapsack problem

Example III, continued

Linear programming dual:

$$\begin{array}{llclcll} \text{minimize} & bu & + & \displaystyle\sum_{j=1}^{n} w_j, & & & \\ \text{subject to} & a_j u & + & w_j & \geq & c_j, & j = 1, \ldots, n, \\ & u & & & \geq & 0, & \\ & & & w_j & \geq & 0, & j = 1, \ldots, n \end{array}$$

Hint: Apply complementarity slackness

Example III, continued: Binary knapsack problem

- Rounding down the fractional variable value yields a feasible solution to (BKP)
- Is it also optimal in (BKP)?

$$\begin{aligned} \text{maximize } & f(\boldsymbol{x}) := 2x_1 + c\,x_2, \\ \text{subject to } & x_1 + c\,x_2 \le c, \qquad (c \in \mathcal{Z}_+) \\ & x_1, x_2 \in \{0,1\}, \end{aligned}$$

- If $c \geq 2$ then $\boldsymbol{x}^* = (0,1)^{\mathrm{T}}$ and $f^* = c$.
- The greedy algorithm, plus rounding, always yields $\bar{\boldsymbol{x}} = (1,0)^{\mathrm{T}}$, with $f(\bar{\boldsymbol{x}}) = 2$
- This solution is arbitrarily bad (when c is large)

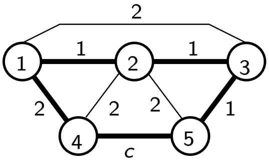
Example IV: The traveling salesperson problem (TSP)

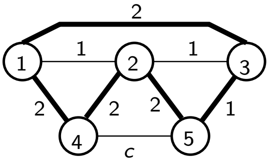
The greedy algorithm for the TSP:

1. Start in node 1
2. Go to the nearest node which is not yet visited
3. Repeat step 2 until no nodes are left
4. Return to node 1; the tour is closed

- Greedy solution

  Not optimal whenever $c > 4$.



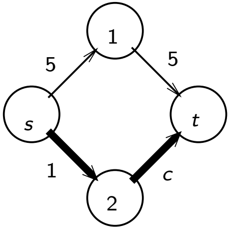


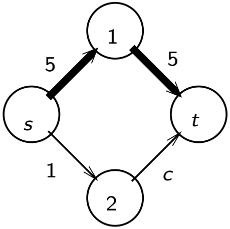
Optimal solution for $c \geq 4$

Example V: the shortest path problem (SPP)

- The greedy algorithm constructs a path that uses – locally – the cheapest link to reach a new node. Optimal?
- Greedy solution

  Not optimal whenever $c > 9$





Optimal solution for $c \geq 9$

Example VI: Semi-matching

$$\begin{aligned} \text{maximize} \quad & f(\boldsymbol{x}) := \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} x_{ij}, \\ \text{subject to} \quad & \sum_{j=1}^{n} x_{ij} \leq 1, \qquad i = 1, \dots, m, \\ & x_{ij} \in \{0, 1\}, \qquad i = 1, \dots, m, \; j = 1, \dots, n. \end{aligned}$$

- Semi-assignment
  Replace maximum $\Longrightarrow$ minimum; "$\leq$" $\Longrightarrow$ "$=$"; let $m = n$
- Algorithm
  For each i:
  1. choose the best (lowest) w_{ij}
  2. Set $x_{ij} = 1$ for that j, and $x_{ij} = 0$ for every other j

Matroid types

- Graph matroid: $\mathcal{F}$ = the set of forests in a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$. *Example problem*: MST
- Partition matroid: Consider a partition of $\mathcal{E}$ into m sets $\mathcal{B}_1, \ldots, \mathcal{B}_m$ and let d_i $(i = 1, \ldots, m)$ be non-negative integers. Let

  $$\mathcal{F} = \{ \mathcal{I} \mid \mathcal{I} \subseteq \mathcal{E}; \quad |\mathcal{I} \cap \mathcal{B}_i| \leq d_i,\ i = 1, \ldots, m \}.$$

  Example problem: semi-matching in bipartite graphs.
- Matrix matroid: $S = (\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ is a set of column vectors and $\mathcal{F}$ is the set of subsets of $\mathcal{E}$ with linearly independent vectors.
- Observe: The above matroids can be expressed as matrix matroids!

Problems over matroid intersections

- Given two matroids $M = (\mathcal{E}, \mathcal{P})$ and $N = (\mathcal{E}, \mathcal{R})$, find the maximum cardinality set in $\mathcal{P} \cap \mathcal{R}$
- Example 1: maximum-cardinality matching in a bipartite graph is the intersection of two partition matroids (with $d_i = 1$). DRAW ILLUSTRATION!
- The intersection of two matroids can *not* generally be solved by using the *greedy algorithm*
- There exist *polynomial algorithms* for them, though
- *Examples*: bipartite matching and assignment problems can be solved as maximum flow problems, which are polynomially solvable

Problems over matroid intersections, cont.

- Example 2: The traveling salesperson problem (TSP) is the intersection of three matroids:
  - one graph matroid
  - two partition matroids

  (formulation on next page: assignment + tree constraints)

- TSP is *not* solvable in polynomial time.

- Conclusion (not proven here):
  - Matroid problems are extremely easy to solve (greedy works)
  - Two-matroid problems are polynomially solvable
  - Three-matroid problems are very difficult (exponential solution time)

- The TSP—different mathematical formulations give rise to different algorithms when Lagrangean relaxed or otherwise decomposed

TSP: Assignment formulation for directed graphs

$$
\begin{array}{lrll}
\text{minimize} & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} & & \\
\text{subject to} & \sum_{j=1}^{n} x_{ij} = 1, & i \in \mathcal{N}, & (1)\\
& \sum_{i=1}^{n} x_{ij} = 1, & j \in \mathcal{N}, & (2)\\
& \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = n, & & (3)\\
& \sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{S}} x_{ij} \le |\mathcal{S}| - 1, & \mathcal{S} \subset \mathcal{N}, & (4)\\
& x_{ij} \in \{0,1\}, & i,j \in \mathcal{N}. &
\end{array}
$$

- (1)–(2): assignment; (3): sum of (1) (redundant); (4): cycle-free
- Relax (3)–(4) $\Rightarrow$ Assignment
- Relax (1)–(2) $\Rightarrow$ 1-MST

TSP in undirected graphs: Node valence formulation

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} && && \\
\text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = 2, && i \in \mathcal{N}, && (1)\\
& \textstyle\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = n, && && (2)\\
& \sum_{(i,j)\in(\mathcal{S},\mathcal{N}\setminus\mathcal{S})} x_{ij} \geq 1, && \mathcal{S} \subset \mathcal{N}, && (3)\\
& x_{ij} \in \{0,1\}, && i,j \in \mathcal{N}. &&
\end{aligned}$$

- (1): valence $= 2$; (2): sum of (1); (3): cycle-free (alt. version)
- Hamiltonian cycle $=$ spanning tree $+$ one link $\Rightarrow$ every node receives valence $= 2$
- Relax (1), except for node s $\Rightarrow$ 1-tree relaxation.
- Relax (3) $\Rightarrow$ 2-matching.