The opportunistic replacement problem—an applications introduction to complexity

Ann-Brith Strömberg

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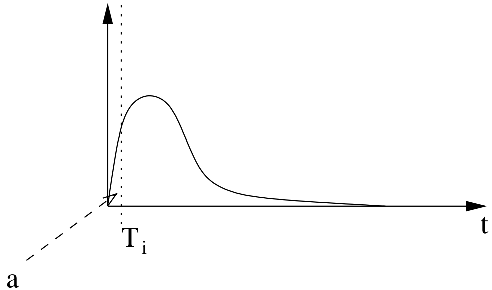
A motivating example

- A wind power turbine with fourteen major components
- A crane is required for the replacement of failed components
- Failure of a component $\Rightarrow$ opportunity for replacement of other components

$\Rightarrow$ Decide whether some other components should also be replaced, based on
  - the components' life distributions (historical and condition data)
  - the price of new components
  - the maintenance occasion (crane) cost
  - the remaining life of the turbine

Deterministic component lives



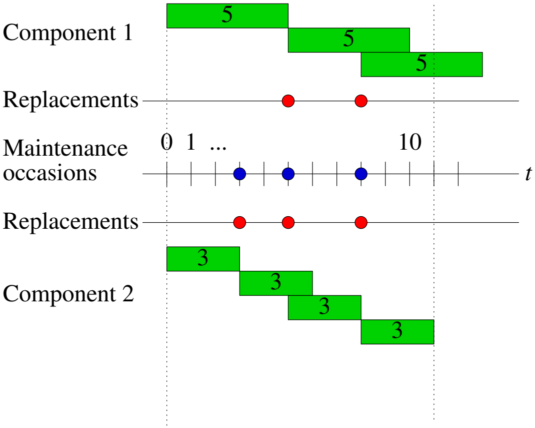
For each component $i \in \mathcal{N}$, choose the component's life T_i such that

$$P(t \leq T_i) = a,$$

for a small value of $a \in (0, 1)$.

The opportunistic replacement problem

cost: $2c_1 + 3c_2 + 3d$



Definition

Given:

- the life T_i of each component i
- costs c_i for replacem. of component i
- costs d for each maintenace occasion
- the time horizon T
- Minimize the total maintenance cost

Modelled as a MILP by Dickman, Epstein & Wilamowsky, 1991

The opportunistic replacement problem

Sets

$$\begin{aligned} \text{Time steps:} \quad & \mathcal{T} = \{1, \dots, T\} \\ \text{Components:} \quad & \mathcal{N} = \{1, \dots, N\} \end{aligned}$$

Variables

$$x_{it} = \begin{cases} 1, & \text{if component } i \text{ is to be replaced at time } t, \\ 0, & \text{otherwise,} \end{cases} \qquad i \in \mathcal{N}, t \in \mathcal{T},$$

$$z_t = \begin{cases} 1, & \text{if maintenance is to be performed at time } t, \\ 0, & \text{otherwise,} \end{cases} \qquad t \in \mathcal{T}.$$

The opportunistic replacement problem

Objective function:

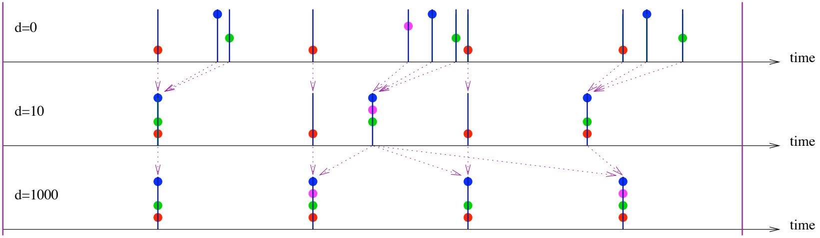
$$\min \quad \sum_{t \in \mathcal{T}} \left(\sum_{i \in \mathcal{N}} c_{it} x_{it} + d z_t \right),$$

Constraints:

$$x_{it} \leq z_t \quad OR \quad \sum_{i \in \mathcal{N}} x_{it} \leq N z_t \ ? \quad 2min!$$

$$\begin{aligned}
\sum_{t=\ell+1}^{\ell+T_i} x_{it} &\geq 1, && \ell \in \{0, \dots, T - T_i\}, \quad i \in \mathcal{N}, \\
x_{it} &\in \{0,1\}, && t \in \mathcal{T}, \quad i \in \mathcal{N}, \\
z_t &\in \{0,1\}, && t \in \mathcal{T}.
\end{aligned}$$

Small example

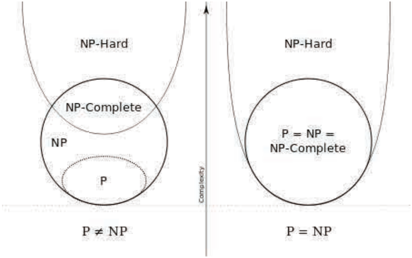


Theoretical properties

- Complexity: NP-hard.
- The integrality requirement on x_{it} can be relaxed
- If the values of z_t are fixed, the resulting optimization problem in the variables x_{it} can be solved using the greedy algorithm
- See Almgren et al. (2012): *"The opportunistic replacement problem: theoretical analyses and numerical tests"*

NP, P, NPC, and NP-hard

- NP: decision problems verifiable in polynomial time
- P: polynomially solvable problems (e.g. shortest path, LP, assignment problem, MST (matroid), matroid intersection)
- NPC: If all problems in NP are polynomially reducible to problem A, then A is in NPC
- NP-hard: If A is NPC and A is polynomially reducible to B then B is NP-hard

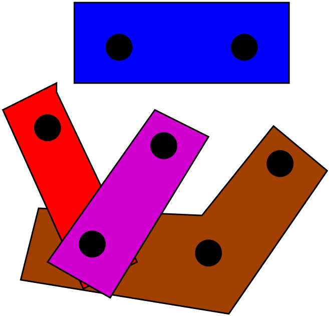


NP, P and NPC

Example (set covering decision problem)

Given: $\mathcal{K} = \{1, \ldots, k\}$, $\mathcal{S}_1, \ldots, \mathcal{S}_m \subset \mathcal{K}$

Question: Is there a cover of cardinality $\leq M$?

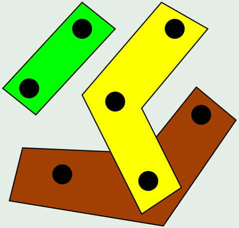


NP-hard

NP-hard: If A is NPC and A is polynomially reducible to B
$\Rightarrow$ then B is NP-hard

Example (set covering optimization problem)

- Given: $\mathcal{K} = \{1, \ldots, k\}$, $\mathcal{S}_1, \ldots, \mathcal{S}_m \subset \mathcal{K}$.
  Question: Which is the cover of lowest cardinality?



The opportunistic replacement problem is NP-hard

Theorem

Set covering is polynomially reducible to the opportunistic replacement problem

Proof.

- Consider the opportunistic replacement problem with $N = k$, $T = m$, $T_i = m$, $d = 1$, and $c_{it} = 0$ if $i \in \mathcal{S}_t$ and $c_{it} = 2$ otherwise
- Show that a solution to this ORP yields an optimal solution to the SC, 10 min!

Total unimodularity

Definition

A matrix A is TU if the determinant of each square submatrix of A is equal to -1,0 or 1.

Example

The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is TU but $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is not TU

Theorem

- *If A is TU then (A, I) is TU.*
- *If A has the "consecutive ones property" then A is TU*
- *If A describes a network flow then A is TU*
- *If A is TU then A^{T} is TU*

Theorem

- *Consider* $x^* \in \arg\min\{c^{\mathrm{T}}x \mid x \in \mathbb{Z}^n, Ax \leq b\}$
- *If A is TU, b is integral and a solution to the LP-relaxation is* $x^*_{LP} \in \arg\min\{c^{\mathrm{T}}x | x \in \mathbb{R}^n, Ax \leq b\}$ *then* $c^{\mathrm{T}}x^* = c^{\mathrm{T}}x^*_{LP}$

Proof.

- Constraint is equivalent to $Ax + Is = b$
- Optimal basis B is a submatrix of (A, I)
- Optimal solution $(x_B, x_N) = (B^{-1}b, 0)$
- Cramers rule $B^{-1} = \frac{B^*}{detB}$, B^* product of elements in B
- B^{-1} is integral $\Rightarrow$ $B^{-1}b$ is integral

$\square$

ORP: Relax integrality on x_{it}

Theorem

If the values of z_t are fixed and integral, then there exists an optimal LP solution to the remaining problem with integral values on x_{it}

Proof.

- Prove this by considering the constraint matrix! 5min!

□

Greedy

Theorem

If the costs depend monotonically on time (i.e., $c_{it} \geq c_{i,t+1}$ or $c_{it} \leq c_{i,t+1}$ and the maintenance occasions are fixed (i.e., z_t are fixed), then a greedy algorithm yields and optimal maintenance schedule

Proof.

Which greedy algorithm? 2 minutes! □

Discussion

Which algorithm could use the TU property and the greedy property?