TMA521/MMA511
Large-Scale Optimization
Lecture 16
Decomposition and solution of a difficult problem—facility location

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Location of facilities which serve customers

Problem settings

- Potential depot sites: $\mathcal{J} = \{1, \dots, n\}$ (geographical locations)
- Existing customers: $\mathcal{I} = \{1, \dots, m\}$ (geographical locations)

  f_j = fixed cost of opening depot (facility) $j \in \mathcal{J}$

  c_{ij} = transportation cost when customer i's demand is fulfilled entirely from depot j ($i \in \mathcal{I}$, $j \in \mathcal{J}$) (cost proportional to amount)

Decision problem

- Which depots to open?
- Which depots to serve which customers, and how much?
- **Goal** minimize cost
- **Assumption:** depots have unlimited capacity (to be removed)

Uncapacitated facility location (UFL)

Variables

$$y_j = \begin{cases} 1, & \text{if depot } j \text{ is opened} \\ 0, & \text{otherwise} \end{cases}$$

$$x_{ij} = \left[\begin{array}{l} \text{proportion of customer } i\text{'s demand} \\ \text{to be delivered from depot } j \end{array} \right]$$

Integer linear optimization model

$$\begin{array}{rlclll} z_0^* = \min & \displaystyle\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j & & & & (0) \\ \text{s.t.} & \displaystyle\sum_{j \in \mathcal{J}} x_{ij} & = & 1, & i \in \mathcal{I} & (1) \\ & x_{ij} - y_j & \leq & 0, & i \in \mathcal{I},\ j \in \mathcal{J} & (2) \\ & x_{ij} & \in & [0, 1], & i \in \mathcal{I},\ j \in \mathcal{J} & (3) \\ & y_j & \in & \{0, 1\}, & j \in \mathcal{J} & (4) \end{array}$$

The mathematical model

$$
\boxed{
\begin{array}{llllll}
z_0^* = \min & \displaystyle\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} c_{ij}x_{ij} + \sum_{j\in\mathcal{J}} f_j y_j & & & & (0)\\
\text{s.t.} & \displaystyle\sum_{j\in\mathcal{J}} x_{ij} & = & 1, & i\in\mathcal{I} & (1)\\
& x_{ij} - y_j & \leq & 0, & i\in\mathcal{I},\ j\in\mathcal{J} & (2)\\
& x_{ij} & \in & [0,1], & i\in\mathcal{I},\ j\in\mathcal{J} & (3)\\
& y_j & \in & \{0,1\}, & j\in\mathcal{J} & (4)
\end{array}
}
$$

(0) Minimize cost

(1) Deliver precisely the demand

(2) Deliver from open depots only

(3) x_{ij} is the *proportion* of the demand of customer i to be delivered from depot j

(4) A depot may *not be partially* opened

Assume that the depots possess limited capacity

- d_i = demand of customer i ($D = \sum_{i \in \mathcal{I}} d_i$)
- b_j = capacity of depot j—if it is opened

  Constraints:

$$\sum_{i \in \mathcal{I}} d_i x_{ij} \leq b_j y_j, \quad j \in \mathcal{J} \quad (5) \qquad (\Rightarrow x_{ij} \leq y_j, \ \forall i, j)$$

$\Rightarrow$ replace (2) (i.e., $x_{ij} \leq y_j$) by (5) $\Rightarrow$

$$\boxed{\begin{array}{rrcllr} z_0^* = \min & \displaystyle\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j & & & & (0) \\ \text{s.t.} & \displaystyle\sum_{j \in \mathcal{J}} x_{ij} & = & 1, & i \in \mathcal{I} & (1) \\ & \displaystyle\sum_{i \in \mathcal{I}} d_i x_{ij} - b_j y_j & \leq & 0, & j \in \mathcal{J} & (5) \\ & x_{ij} & \in & [0, 1], & i \in \mathcal{I}, j \in \mathcal{J} & (3) \\ & y_j & \in & \{0, 1\}, & j \in \mathcal{J} & (4) \end{array}}$$

Capacitated facility location (CFL)

$$
\begin{aligned}
z^* = \min \quad & \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} c_{ij}x_{ij} + \sum_{j\in\mathcal{J}} f_j y_j & & & & (0)\\
\text{s.t.} \quad & \sum_{j\in\mathcal{J}} x_{ij} & = \; & 1, & i\in\mathcal{I} & \quad (1)\\
& \sum_{i\in\mathcal{I}} d_i x_{ij} - b_j y_j & \leq \; & 0, & j\in\mathcal{J} & \quad (5)\\
& x_{ij} & \in \; & [0,1], & i\in\mathcal{I},\ j\in\mathcal{J} & \quad (3)\\
& y_j & \in \; & \{0,1\}, & j\in\mathcal{J} & \quad (4)
\end{aligned}
$$

Observation: The total capacity of open depots must cover the entire demand $\Longrightarrow$ an additional (redundant) constraint:

$$
(1),(5) \Rightarrow \overbrace{\sum_{j\in\mathcal{J}} b_j y_j}^{\text{capacity}} \geq \sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}} d_i x_{ij} = \sum_{i\in\mathcal{I}} d_i \sum_{j\in\mathcal{J}} x_{ij} = \sum_{i\in\mathcal{I}} d_i \cdot 1 = \overbrace{D}^{\text{demand}} \quad (6)
$$

Add this constraint to the model

Resulting integer linear optimization model

$$
\begin{aligned}
z^* = \min \quad & \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{ij} x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j && && \\
\text{s.t.} \quad & \sum_{j \in \mathcal{J}} x_{ij} = 1, && i \in \mathcal{I} && (1) \\
& \sum_{i \in \mathcal{I}} d_i x_{ij} - b_j y_j \leq 0, && j \in \mathcal{J} && (5) \\
& \sum_{j \in \mathcal{J}} b_j y_j \geq D, && && (6) \\
& x_{ij} \in [0,1], && i \in \mathcal{I},\ j \in \mathcal{J} && (3) \\
& y_j \in \{0,1\}, && j \in \mathcal{J} && (4)
\end{aligned}
$$

A direct Benders decomposition, I

- Define

$$S := \left\{ y \in \{0,1\}^{|\mathcal{J}|} \;\middle|\; \sum_{j\in\mathcal{J}} b_j y_j \geq D \right\} \quad \text{(a 0/1 kapsack constraint}$$

- The resulting decomposition:

$$z^* = \min_{y\in S} \left(\sum_{j\in\mathcal{J}} f_j y_j + \begin{bmatrix} \min\limits_{x\geq 0} & \sum\limits_{i\in\mathcal{I}}\sum\limits_{j\in\mathcal{J}} c_{ij}x_{ij} & & & \\ \text{s.t.} & \sum\limits_{j\in\mathcal{J}} x_{ij} & = & 1, & i\in\mathcal{I} \\ & \sum\limits_{i\in\mathcal{I}} d_i x_{ij} & \leq & b_j y_j, & j\in\mathcal{J} \end{bmatrix} \right)$$

- E.g., optimization under uncertainty, scenario ℓ: $c_{ij\ell} \Rightarrow x_{ij\ell}$

A direct Benders decomposition, II

- LP dual of Benders subproblem (is bounded since the primal subproblem is feasible for all $y \in S$):

$$
\begin{array}{lllll}
\max\limits_{u,v} & \sum\limits_{i \in \mathcal{I}} u_i & -\sum_{j \in \mathcal{J}} (b_j y_j) v_j & & \\
\text{s.t.} & u_i & -d_i v_j & \leq & c_{ij}, \quad i \in \mathcal{I}, j \in \mathcal{J} \\
 & & v_j & \geq & 0, \quad j \in \mathcal{J}
\end{array}
$$

- Extreme point solutions: $\left([\hat{u}_i^k]_{i \in \mathcal{I}}, [\hat{v}_j^k]_{j \in \mathcal{J}} \right), \quad k = 1, \ldots, p$
- Benders master problem

$$
\begin{array}{lllll}
z^* = \min\limits_{y,z} & z & & & \\
 & & \sum_{j \in \mathcal{J}} b_j y_j & \geq D & \\
\text{s.t.} & z & + \sum_{j \in \mathcal{J}} (b_j \hat{v}_j^k - f_j) y_j & \geq \sum_{i \in \mathcal{I}} \hat{u}_i^k, & k = 1, \ldots, p \\
 & & y_j & \in \{0,1\}, & j \in \mathcal{J}
\end{array}
$$

Trick: variable splitting

- Replace x_{ij} by w_{ij} in constraint (1) and in "half" the objective
- Let $0 \leq \alpha \leq 1$.
- Add the constraints $x_{ij} = w_{ij}$

$$
\begin{array}{rrcll}
z^* = \min & \alpha \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} c_{ij}x_{ij} & + & (1-\alpha) \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} c_{ij}w_{ij} \quad + \quad \sum_{j\in\mathcal{J}} f_j y_j & \\
\text{s.t.} & \sum_{j\in\mathcal{J}} w_{ij} & = & 1, \qquad i \in \mathcal{I} & (1) \\
& \sum_{i\in\mathcal{I}} d_i x_{ij} - b_j y_j & \leq & 0, \qquad j \in \mathcal{J} & (5) \\
& \sum_{j\in\mathcal{J}} b_j y_j & \geq & D, & (6) \\
& w_{ij} - x_{ij} & = & 0, \qquad i \in \mathcal{I},\ j \in \mathcal{J} & (7) \\
& x_{ij} & \in & [0,1], \qquad i \in \mathcal{I},\ j \in \mathcal{J} & (3) \\
& w_{ij} & \geq & 0, \qquad i \in \mathcal{I},\ j \in \mathcal{J} & (8) \\
& y_j & \in & \{0,1\}, \qquad j \in \mathcal{J} & (4)
\end{array}
$$

Lagrangian relaxation

- The constraints (7) tie together the variables $(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{w}$
- Lagrangian relax these with multipliers λ_{ij}

$\Longrightarrow$ Lagrange function

$$
\begin{aligned}
&L(\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{y}, \boldsymbol{\lambda}) = \\
&= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left[\alpha c_{ij} x_{ij} + (1-\alpha) c_{ij} w_{ij} + \overbrace{\lambda_{ij}(w_{ij} - x_{ij})}^{\text{penalty}} \right] + \sum_{j \in \mathcal{J}} f_j y_j \\
&= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} (\alpha c_{ij} - \lambda_{ij}) x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \left[(1-\alpha) c_{ij} + \lambda_{ij} \right] w_{ij}
\end{aligned}
$$

- For a fixed value of $\boldsymbol{\lambda}$:
  Minimize the Lagrange function under the constraints (1), (5), (6), (3), (8) & (4)

The Lagrangian subproblem

The constraints (7) are "moved" to the objective, the rest are rearranged:

$$q(\boldsymbol{\lambda}) = \min \quad \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} (\alpha c_{ij} - \lambda_{ij}) x_{ij} + \sum_{j \in \mathcal{J}} f_j y_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} [(1-\alpha) c_{ij} + \lambda_{ij}] w_{ij}$$

$$\begin{aligned}
\text{s.t.} \quad \sum_{i \in \mathcal{I}} d_i x_{ij} - b_j y_j &\leq 0, && j \in \mathcal{J} && (5)\\
\sum_{j \in \mathcal{J}} b_j y_j &\geq D, && && (6)\\
x_{ij} &\in [0,1], && i \in \mathcal{I},\ j \in \mathcal{J} && (3)\\
y_j &\in \{0,1\}, && j \in \mathcal{J} && (4)\\
\sum_{j \in \mathcal{J}} w_{ij} &= 1, && i \in \mathcal{I} && (1)\\
w_{ij} &\geq 0, && i \in \mathcal{I},\ j \in \mathcal{J} && (8)
\end{aligned}$$

Separates into one problem in $(\boldsymbol{x}, \boldsymbol{y})$ and $|\mathcal{I}|$ problems in $\boldsymbol{w}$

The subproblem in $\mathbf{x}$ and $\mathbf{y}$ (for a fixed value of $\boldsymbol{\lambda}$)

$$
\begin{aligned}
q_{\mathbf{xy}}(\boldsymbol{\lambda}) = \min_{x,y} \quad & \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} [\alpha c_{ij} - \lambda_{ij}] x_{ij} + \sum_{j\in\mathcal{J}} f_j y_j \\
\text{s.t.} \quad & \sum_{j\in\mathcal{J}} b_j y_j \geq D, && (6) \\
& \sum_{i\in\mathcal{I}} d_i x_{ij} \leq b_j y_j, \quad j\in\mathcal{J} && (5) \\
& x_{ij} \in [0,1], \quad i\in\mathcal{I},\ j\in\mathcal{J} && (3) \\
& y_j \in \{0,1\}, \quad j\in\mathcal{J} && (4)
\end{aligned}
$$

- This problem can be further decomposed through the following observation
- For every solution $\boldsymbol{y}$ (such that $\sum_{j\in\mathcal{J}} b_j y_j \geq D$ holds) we have the following:
  - If $y_j = 0$ then $x_{ij} = 0$, $i \in \mathcal{I}$ must hold
  - If $y_j = 1$ then $\sum_{i\in\mathcal{I}} d_i x_{ij} \leq b_j$ and $x_{ij} \in [0,1]$ must hold

The value/cost of opening depot j, i.e., letting $y_j = 1$ (in the $(\boldsymbol{x}, \boldsymbol{y})$-subproblem)

- Solve $|\mathcal{J}|$ continuous knapsack problems (easy)

$$
[\mathrm{CKSP}_j] \qquad v_j(\boldsymbol{\lambda}) = f_j + \min_{\mathbf{x}} \quad \sum_{i \in \mathcal{I}} [\alpha c_{ij} - \lambda_{ij}]\, x_{ij}
$$
$$
\text{s.t.} \quad \sum_{i \in \mathcal{I}} d_i x_{ij} \leq b_j
$$
$$
x_{ij} \in [0, 1], \quad i \in \mathcal{I}
$$

- Then, decide which depots to open (for a certain value of $\boldsymbol{\lambda}$)
- Project onto the $\boldsymbol{y}$ space (i.e., Benders) $\Rightarrow$ one 0/1 knapsack problem

$$
[0/1\text{-KSP}] \qquad q_{\mathbf{xy}}(\boldsymbol{\lambda}) = \min_{\mathbf{y}} \quad \sum_{j \in \mathcal{J}} v_j(\boldsymbol{\lambda}) \cdot y_j
$$
$$
\text{s.t.} \quad \sum_{j \in \mathcal{J}} b_j y_j \geq D,
$$
$$
y_j \in \{0, 1\}, \quad j \in \mathcal{J}
$$

Solving the continuous knapsack problems [CKSP_j]

Greedy algorithm

- Sort the values $\frac{\alpha c_{ij} - \lambda_{ij}}{d_i} < 0$, $i \in \mathcal{I}$, in increasing order
  $\Rightarrow$ indices $\{i_1, i_2, \ldots, i_p\}$, where $p \leq m = |\mathcal{I}|$
- $\bar{x}_{ij} := 0$, $i \in \mathcal{I}$, $k := 0$

  repeat

  $$k := k + 1$$
  $$\bar{x}_{i_k j} := \min \left\{ 1; \left(b_j - \sum_{s=1}^{k-1} d_{i_s} \bar{x}_{i_s j} \right) / d_{i_k} \right\}$$

  until $\sum_{s=1}^{k} d_i \bar{x}_{i_s j} = b_j$ or $k = p$
- The solution fulfills $\sum_{i \in \mathcal{I}} d_i \bar{x}_{ij} \leq b_j$ and $\bar{x}_{ij} \in [0, 1]$, $i \in \mathcal{I}$

$\Rightarrow$ **The value/cost of opening depot j**

- $$v_j(\boldsymbol{\lambda}) = f_j + \sum_{k=1}^{p} \sum_{j \in \mathcal{J}} \left[\alpha c_{i_k j} - \lambda_{i_k j} \right] \bar{x}_{i_k j}$$

Solving the 0/1 knapsack problems [0/1-KSP]

$$
\begin{aligned}
q_{\mathbf{xy}}(\boldsymbol{\lambda}) = \min_{\mathbf{y}} \quad & \sum_{j \in \mathcal{J}} v_j(\boldsymbol{\lambda}) \cdot y_j \\
\text{s.t.} \quad & \sum_{j \in \mathcal{J}} b_j y_j \geq D, \\
& y_j \in \{0, 1\}, \quad j \in \mathcal{J}
\end{aligned}
$$

where $v_j(\boldsymbol{\lambda}) = f_j + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} [\alpha c_{ij} - \lambda_{ij}] \bar{x}_{ij}$
and $\bar{x}_{ij}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$, are computed by the greedy algorithm

- 0/1-KSP *cannot* be solved in polynomial time
- Solve using Dynamic Programming or Branch & Bound (e.g., CPLEX)

Summary of the solution of the $(\mathbf{x}, \mathbf{y})$-problem

For a fixed value of the penalty $\boldsymbol{\lambda}$ (multiplier)

- Solve $|\mathcal{J}|$ continuous knapsack problems

$\Rightarrow$ Solution $\bar{x}_{ij}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$ (greedy)

$\Rightarrow$ The value of opening depot j: $v_j(\boldsymbol{\lambda})$, $j \in \mathcal{J}$

- Solve a 0/1-knapsack problem

$\Rightarrow$ $y_j(\boldsymbol{\lambda}) \in \{0, 1\}$, $j \in \mathcal{J}$

$\Rightarrow$ $x_{ij}(\boldsymbol{\lambda}) = \bar{x}_{ij} \cdot y_j(\boldsymbol{\lambda})$, $i \in \mathcal{I}$, $j \in \mathcal{J}$

- Solution $(\boldsymbol{x}(\boldsymbol{\lambda}), \boldsymbol{y}(\boldsymbol{\lambda}))$

The subproblem in $\mathbf{w}$ (for a fixed value of $\boldsymbol{\lambda}$)

$|\mathcal{I}|$ semi-assignment problems (SAP) (partition matroid)

$$q_{\mathbf{w}}(\boldsymbol{\lambda}) = \sum_{i \in \mathcal{I}} \left[\begin{array}{ll} \min\limits_{\mathbf{w}} & \sum\limits_{j \in \mathcal{J}} \left[(1-\alpha) c_{ij} + \lambda_{ij} \right] w_{ij} \\ \text{s.t.} & \sum\limits_{j \in \mathcal{J}} w_{ij} = 1, \quad w_{ij} \geq 0, \quad j \in \mathcal{J} \end{array} \right]$$

Solve the semi-assignment problem $i \in \mathcal{I}$

- $\ell_i := \arg\min\limits_{j \in \mathcal{J}} \{(1-\alpha) c_{ij} + \lambda_{ij}\}$
- $w_{i\ell_i}(\boldsymbol{\lambda}) := 1,$
- $w_{ij}(\boldsymbol{\lambda}) := 0,\ j \neq \ell_i$

The value of the relaxed problem for a fixed value of $\boldsymbol{\lambda}$

$$q(\boldsymbol{\lambda}) \quad = \quad \underbrace{q_{\mathsf{xy}}(\boldsymbol{\lambda})}_{\text{difficult}} \quad + \quad \underbrace{q_{\mathsf{w}}(\boldsymbol{\lambda})}_{\text{simple}}$$

- It holds that $q(\boldsymbol{\lambda}) \leq z^*$ for all $\boldsymbol{\lambda} \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}$ (weak duality)
- λ_{ij} is the penalty for violating the constraint $w_{ij} = x_{ij}$
- Find best *underestimate* of $z^* \iff$ find optimal values for the penalties λ_{ij}
- Weak duality: $\boxed{q^* := \max_{\boldsymbol{\lambda} \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}} q(\boldsymbol{\lambda}) \leq z^*}$
- Not strong duality: $\boxed{q^* < z^*}$

How to find better values for λ_{ij}? Subgradient optimization.

Penalty: min $\quad \ldots \quad + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \lambda_{ij}(w_{ij} - x_{ij})$

- If $w_{ij}(\boldsymbol{\lambda}) > x_{ij}(\boldsymbol{\lambda}) \Rightarrow$ Increase the value of λ_{ij} (higher penalty for violating the constraint)
- If $w_{ij}(\boldsymbol{\lambda}) < x_{ij}(\boldsymbol{\lambda}) \Rightarrow$ Decrease the value of λ_{ij} (higher penalty for violating the constraint)
- **Subgradient algorithm** to find optimal penalties $\boldsymbol{\lambda}^*$

$$\lambda_{ij}^{t+1} = \lambda_{ij}^{t} + \rho_t \left[w_{ij}(\boldsymbol{\lambda}^t) - x_{ij}(\boldsymbol{\lambda}^t) \right], \qquad t = 0, 1, \ldots$$

  where $\rho_t > 0$ is a step length, decreasing with t
- Use **feasibility heuristic** from every $\left[\boldsymbol{x}(\boldsymbol{\lambda}^t), \boldsymbol{w}(\boldsymbol{\lambda}^t), \boldsymbol{y}(\boldsymbol{\lambda}^t) \right]$ to yield a **feasible solution** to CFL
- E.g., open more depots, send only from open depots, $\boldsymbol{x} := \boldsymbol{w}, \ldots$

Example: $|\mathcal{I}| = 4$, $|\mathcal{J}| = 3$, $\alpha = \frac{1}{2}$

$$(c_{ij}) = \begin{pmatrix} 6 & 2 & 4 \\ 2 & 8 & 4 \\ 16 & 2 & 6 \\ 10 & 12 & 4 \end{pmatrix}, (f_j) = \begin{pmatrix} 11 \\ 16 \\ 21 \end{pmatrix}, (d_i) = \begin{pmatrix} 6 \\ 4 \\ 8 \\ 5 \end{pmatrix}, (b_j) = \begin{pmatrix} 12 \\ 10 \\ 13 \end{pmatrix}$$

The 0/1-knapsack problem

$$\begin{array}{rl} q_{\mathbf{xy}}(\boldsymbol{\lambda}) = \min & \sum_{j=1}^{3} v_j(\boldsymbol{\lambda}) \cdot y_j \\ \text{s.t.} & 12y_1 + 10y_2 + 13y_3 \geq 23 \\ & \boldsymbol{y} \in \{0,1\}^3 \end{array}$$

Let $(\lambda_{ij}^t) = \begin{pmatrix} 7 & 0 & 0 \\ 3 & 10 & 2 \\ 5 & 2 & 0 \\ 0 & 7 & 5 \end{pmatrix}$

Observe: $y_3 = 1$ must hold (why?)

$$\underbrace{\ldots \Longrightarrow \ldots}_{\text{next page}} \qquad \begin{array}{rl} q_{\mathbf{xy}}(\boldsymbol{\lambda}^t) = \min & 5y_1 + 8.875y_2 + 18y_3 \\ \text{s.t.} & 12y_1 + 10y_2 + 13y_3 \geq 23, \quad \boldsymbol{y} \in \{0,1\}^3 \end{array}$$

The value of opening a depot

$$v_1(\boldsymbol{\lambda}^t) = 11 + \begin{array}{ll} \min & -4x_{11} - 2x_{21} + 3x_{31} + 5x_{41} \\ \text{s.t.} & 6x_{11} + 4x_{21} + 8x_{31} + 5x_{41} \leq 12, \quad \boldsymbol{x}_{\cdot 1} \in [0,1]^4 \end{array}$$

$$\Rightarrow \boxed{\text{solution } \bar{x}_{11} = \bar{x}_{21} = 1,\ \bar{x}_{31} = \bar{x}_{41} = 0, \quad v_1(\boldsymbol{\lambda}^t) = 5}$$

$$v_2(\boldsymbol{\lambda}^t) = 16 + \begin{array}{ll} \min & x_{12} - 6x_{22} - x_{32} - x_{42} \\ \text{s.t.} & 6x_{12} + 4x_{22} + 8x_{32} + 5x_{42} \leq 10, \quad \boldsymbol{x}_{\cdot 2} \in [0,1]^4 \end{array}$$

$$\Rightarrow \boxed{\text{solution } \bar{x}_{22} = \bar{x}_{42} = 1,\ \bar{x}_{32} = \tfrac{1}{8},\ \bar{x}_{12} = 0, \quad v_2(\boldsymbol{\lambda}^t) = 8.875}$$

$$v_3(\boldsymbol{\lambda}^t) = 21 + \begin{array}{ll} \min & 2x_{13} + 0x_{23} + 3x_{33} - 3x_{43} \\ \text{s.t.} & 6x_{13} + 4x_{23} + 8x_{33} + 5x_{43} \leq 13, \quad \boldsymbol{x}_{\cdot 3} \in [0,1]^4 \end{array}$$

$$\Rightarrow \boxed{\text{solution } \bar{x}_{23} = \bar{x}_{43} = 1,\ \bar{x}_{13} = \bar{x}_{33} = 0, \quad v_3(\boldsymbol{\lambda}^t) = 18}$$

The solution to the $(\boldsymbol{x}, \boldsymbol{y})$-problem for $\boldsymbol{\lambda} = \boldsymbol{\lambda}^t$

- Open depots: $\boldsymbol{y}(\boldsymbol{\lambda}^t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

- Transport goods from open depots to customers (this solution does *not* fulfil the demand constraints (1) for each customer):
$$\boldsymbol{x}(\boldsymbol{\lambda}^t) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Objective value: $q_{\mathbf{xy}}(\boldsymbol{\lambda}^t) = 5 + 0 + 18 = 23$

The $\boldsymbol{w}$-problem separates into one problem for each customer i

$$\boxed{q_{\mathbf{w}}(\boldsymbol{\lambda}^t) = \sum_{i=1}^{4} q_{\mathbf{w}}^i(\boldsymbol{\lambda}^t)}$$

where

$$\begin{aligned} q_{\mathbf{w}}^i(\boldsymbol{\lambda}^t) = \min_{w} \quad & \sum_{j=1}^{3} \left[(1-\alpha)c_{ij} + \lambda_{ij}^t\right] w_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{3} w_{ij} = 1, \quad w_{ij} \geq 0, \; j = 1,2,3 \end{aligned}$$

and $1 - \alpha = \frac{1}{2}$

The solution to the $\boldsymbol{w}$-problem

$$q^1_{\mathbf{w}}(\boldsymbol{\lambda}^t) = \quad \min \quad 10w_{11} + w_{12} + 2w_{13}$$
$$\text{s.t.} \quad w_{11} + w_{12} + w_{13} = 1, \quad w_{1j} \geq 0, \ j = 1, 2, 3$$

$$\boxed{\Rightarrow \text{solution } w_{12}(\boldsymbol{\lambda}^t) = 1, \ w_{11}(\boldsymbol{\lambda}^t) = w_{13}(\boldsymbol{\lambda}^t) = 0, \ q^1_{\mathbf{w}}(\boldsymbol{\lambda}^t) = 1}$$

$$q^2_{\mathbf{w}}(\boldsymbol{\lambda}^t) = \quad \min \quad 4w_{21} + 14w_{22} + 4w_{23}$$
$$\text{s.t.} \quad w_{21} + w_{22} + w_{23} = 1, \quad w_{2j} \geq 0, \ j = 1, 2, 3$$

$$\boxed{\Rightarrow \text{solution } w_{21}(\boldsymbol{\lambda}^t) = 1, \ w_{22}(\boldsymbol{\lambda}^t) = w_{23}(\boldsymbol{\lambda}^t) = 0, \ q^2_{\mathbf{w}}(\boldsymbol{\lambda}^t) = 4}$$

$$q^3_{\mathbf{w}}(\boldsymbol{\lambda}^t) = \quad \min \quad 13w_{31} + 3w_{32} + 3w_{33}$$
$$\text{s.t.} \quad w_{31} + w_{32} + w_{33} = 1, \quad w_{3j} \geq 0, \ j = 1, 2, 3$$

$$\boxed{\Rightarrow \text{solution } w_{32}(\boldsymbol{\lambda}^t) = w_{33}(\boldsymbol{\lambda}^t) = \tfrac{1}{2}, \ w_{31}(\boldsymbol{\lambda}^t) = 0, \ q^3_{\mathbf{w}}(\boldsymbol{\lambda}^t) = 3}$$

$$q^4_{\mathbf{w}}(\boldsymbol{\lambda}^t) = \quad \min \quad 5w_{41} + 13w_{42} + 7w_{43}$$
$$\text{s.t.} \quad w_{41} + w_{42} + w_{43} = 1, \quad w_{4j} \geq 0, \ j = 1, 2, 3$$

$$\boxed{\Rightarrow \text{solution } w_{41}(\boldsymbol{\lambda}^t) = 1, \ w_{42}(\boldsymbol{\lambda}^t) = w_{43}(\boldsymbol{\lambda}^t) = 0, \ q^4_{\mathbf{w}}(\boldsymbol{\lambda}^t) = 5}$$

The solution to the (x, y)- and w-problems

- Send the right amout of goods to each customer (this solution presumes that *all* depots are opened):

$$\boldsymbol{w}(\boldsymbol{\lambda}^t) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

- Objective value: $q_{\mathbf{w}}(\boldsymbol{\lambda}^t) = 13$
- Total objective value $q(\boldsymbol{\lambda}^t) = q_{\mathbf{xy}}(\boldsymbol{\lambda}^t) + q_{\mathbf{w}}(\boldsymbol{\lambda}^t) = 35$
- Lower bound on the optimal objective value: $z^* \geq 35$

Compute a new λ-vector (here, the steplength $\rho_t = 8$)

$$\begin{aligned} \boldsymbol{\lambda}^{t+1} &= \boldsymbol{\lambda}^t + \rho_t \left[\boldsymbol{w}(\boldsymbol{\lambda}^t) - \boldsymbol{x}(\boldsymbol{\lambda}^t) \right] \\ &= \begin{pmatrix} 7 - \rho_t & \rho_t & 0 \\ 3 & 10 & 2 - \rho_t \\ 5 & 2 + \frac{\rho_t}{2} & \frac{\rho_t}{2} \\ \rho_t & 7 & 5 - \rho_t \end{pmatrix} = \begin{pmatrix} -1 & 8 & 0 \\ 3 & 10 & -6 \\ 5 & 6 & 4 \\ 8 & 7 & -3 \end{pmatrix} \end{aligned}$$

Feasible solution $\Leftrightarrow \boldsymbol{x}(\boldsymbol{\lambda}^t) = \boldsymbol{w}(\boldsymbol{\lambda}^t)$? If not $\Rightarrow$ Feasibility heuristic

- Open the depots given by $\boldsymbol{y}(\boldsymbol{\lambda}^t) \Rightarrow \boldsymbol{y}^H = \boldsymbol{y}(\boldsymbol{\lambda}^t) = (1, 0, 1)^{\mathrm{T}}$
- Transport goods only from opened depots:
  $y_j^H = 0 \Rightarrow x_{ij}^H = 0,\ i \in \mathcal{I}$
- Fulfill the demand but do not violate the capacity restrictions

  Let $\boldsymbol{x}^H = \begin{pmatrix} \frac{1}{6} & 0 & \frac{5}{6} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

  $\Rightarrow z^H = 6 \cdot \frac{1}{6} + 4 \cdot \frac{5}{6} + 2 + 6 + 10 + 11 + 21 = 52 + \frac{1}{3}$

  $\Rightarrow z^* \in [35, 52 + \frac{1}{3}] = [q(\boldsymbol{\lambda}^t), z^H]$

More about the solution method

- Choice of step lengths (ρ_t): subgradient optimization, convergence to an optimal value of $\boldsymbol{\lambda}$
- Feasibility heuristics can be made more or less sophisticated
- More ways to Lagrangian relax continuous constraints in an optimization model:
  - E.g.: Lagrangian relax (1) *or* (5) (with multipliers $\mu_i \in \mathbb{R}$ and $\nu_j \in \mathbb{R}_+$, respectively) in the original formulation (CFL)