TMA521/MMA511
Large Scale Optimization
Lecture 7
Cutting plane methods and column generation

Ann-Brith Strömberg

31 Januari 2018

A standard LP problem and its Lagrangian dual

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v_{\mathrm{LP}} := \quad \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\top \boldsymbol{x}, \\
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$$X := \left\{ \boldsymbol{x} \in \mathbb{R}^n_+ \,\middle|\, \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \right\}$$

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  is bounded
- Let $\left\{\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^K\right\}$ be the set of extreme points in X
- It follows that $X = \operatorname{conv}\left\{\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^K\right\}$

The Lagrangian dual

- The Lagrangian dual of the LP with respect to relaxing the constraints $\boldsymbol{Dx} \leq \boldsymbol{d}$ is

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- The Lagrangian dual function is defined as:

$$q(\boldsymbol{\mu}) := \min_{\mathbf{x} \in X} \left\{ \boldsymbol{c}^\top \boldsymbol{x} + \boldsymbol{\mu}^\top (\boldsymbol{Dx} - \boldsymbol{d}) \right\} \tag{1a}$$

$$= \min_{i \in \{1,\ldots,K\}} \left\{ \boldsymbol{c}^\top \boldsymbol{x}^i + \boldsymbol{\mu}^\top (\boldsymbol{Dx}^i - \boldsymbol{d}) \right\} \tag{1b}$$

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- Solution set to the subproblem at $\boldsymbol{\mu}$:

$$X(\boldsymbol{\mu}) := \operatorname*{argmin}_{\mathbf{x} \in X} \left\{ \boldsymbol{c}^{\top} \boldsymbol{x} + \boldsymbol{\mu}^{\top} (\boldsymbol{Dx} - \boldsymbol{d}) \right\}$$

$$= \operatorname{conv} \left\{ \operatorname*{argmin}_{\mathbf{x}^{i} : i = 1, \ldots, K} \left\{ \boldsymbol{c}^{\top} \boldsymbol{x}^{i} + \boldsymbol{\mu}^{\top} (\boldsymbol{Dx}^{i} - \boldsymbol{d}) \right\} \right\}$$

An equivalent formulation of the Lagrangian dual

- Due to (1), for all $\boldsymbol{\mu} \geq \mathbf{0}$ the function q fulfills the inequalities
$$q(\boldsymbol{\mu}) \leq \boldsymbol{c}^\top \boldsymbol{x}^i + \boldsymbol{\mu}^\top (\boldsymbol{D}\boldsymbol{x}^i - \boldsymbol{d}), \qquad i = 1, \ldots, K.$$

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- If, at an optimal dual solution $\boldsymbol{\mu}^*$, the solution set $X(\boldsymbol{\mu}^*)$ is a singleton, i.e., $X(\boldsymbol{\mu}^*) = \{\boldsymbol{x}^*\}$, then $\boldsymbol{x}^*$ is optimal (and unique) — thanks to strong duality
- This typically does not happen, unless an optimal solution $\boldsymbol{x}^*$ happens to be an extreme point of X
- But $\boldsymbol{x}^*$ can always be expressed as a convex combination of extreme points of X (cf. the previous lecture)

A cutting plane method for the Lagrangian dual problem

- Suppose that only a subset, $\{\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^k\}$ of the extreme points $\{\boldsymbol{x}^1, \boldsymbol{x}^2, \ldots, \boldsymbol{x}^K\}$ is known (hence, $k \leq K$), and consider the following relaxation of the Lagrangian dual problem:

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(\boldsymbol{\mu}^k, z^k) \in \operatorname*{argmax}_{\mu, z} \; & z, && \text{(2a)}\\
\text{subject to } & z \leq \boldsymbol{c}^\top \boldsymbol{x}^i + \boldsymbol{\mu}^\top (\boldsymbol{D}\boldsymbol{x}^i - \boldsymbol{d}), \; i = 1, \ldots, k, && \text{(2b)}\\
& \boldsymbol{\mu} \geq \boldsymbol{0} && \text{(2c)}
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- For $k = 1, \dots, K$ it holds that $z^k \geq v_{\mathrm{L}}$
- How do we determine whether an optimal solution to the Lagrangian dual is found?
- If it holds that

$$z^k \leq \boldsymbol{c}^\top \boldsymbol{x}^i + (\boldsymbol{\mu}^k)^\top (\boldsymbol{D}\boldsymbol{x}^i - \boldsymbol{d}), \quad i = 1, \dots, K,$$

then $\boldsymbol{\mu}^k$ is optimal in the dual. *Why?*

Check for optimality—generate new inequality

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$$q(\boldsymbol{\mu}^k) := \min_{\mathbf{x}\in X} \left\{ \boldsymbol{c}^\top \boldsymbol{x} + (\boldsymbol{\mu}^k)^\top (\boldsymbol{D}\boldsymbol{x} - \boldsymbol{d}) \right\} \quad (3)$$
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- Otherwise, we have identified a constraint of the form

$$z \leq \boldsymbol{c}^\top \boldsymbol{x}^{k+1} + \boldsymbol{\mu}^\top (\boldsymbol{D}\boldsymbol{x}^{k+1} - \boldsymbol{d}), \tag{4}$$

which is violated at $(\boldsymbol{\mu}^k, z^k)$

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  - The latter means that the following (strict) inequality holds:

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- Add the inequality (4) to (2b), update $k := k + 1$, and resolve the LP (2)

Cutting plane algorithm

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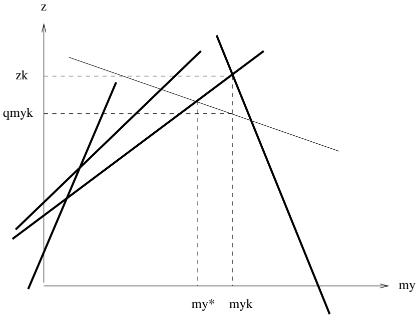
- It is based on the addition of constraints to the dual problem, in order to improve the solution, in the process of *cutting off* the previous (optimal) point
- The **thick** lines correspond to the subset of k inequalities known at iteration k



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- The resulting maximization will then yield the optimal solution μ^*, as illustrated in the figure

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- The resulting maximization will then yield the optimal solution μ^*, as illustrated in the figure
- How do we generate an optimal primal solution from this scheme?
  - Let us look at the LP dual of the Lagrangian dual problem (2) in this cutting plane algorithm

Duality relations and the Dantzig–Wolfe algorithm

- We rewrite the relaxed Lagrangian dual problem (2) as

$$
\begin{aligned}
z^k := \quad & \max_{(z,\boldsymbol{\mu})} \quad z, \\
& \text{subject to} \quad z - \boldsymbol{\mu}^\top(\boldsymbol{D}\boldsymbol{x}^i - \boldsymbol{d}) \le \boldsymbol{c}^\top \boldsymbol{x}^i, \quad i = 1, \dots, k, \\
& \qquad\qquad\qquad\qquad \boldsymbol{\mu} \ge \mathbf{0}
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- Defining the LP dual variables $\lambda_i \geq 0$, $i = 1, \dots, k$, we obtain the LP dual:

$$
\begin{aligned}
z^k = \quad & \min \quad \textstyle\sum_{i=1}^k (\boldsymbol{c}^\top \boldsymbol{x}^i)\lambda_i, \\
& \text{subject to} \quad \textstyle\sum_{i=1}^k \lambda_i = 1, \\
& \qquad -\textstyle\sum_{i=1}^k (\boldsymbol{D}\boldsymbol{x}^i - \boldsymbol{d})\lambda_i \geq \boldsymbol{0}, \\
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The linear programming dual rewritten

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& \qquad \lambda_i \geq 0, \qquad i = 1, \ldots, k, \\
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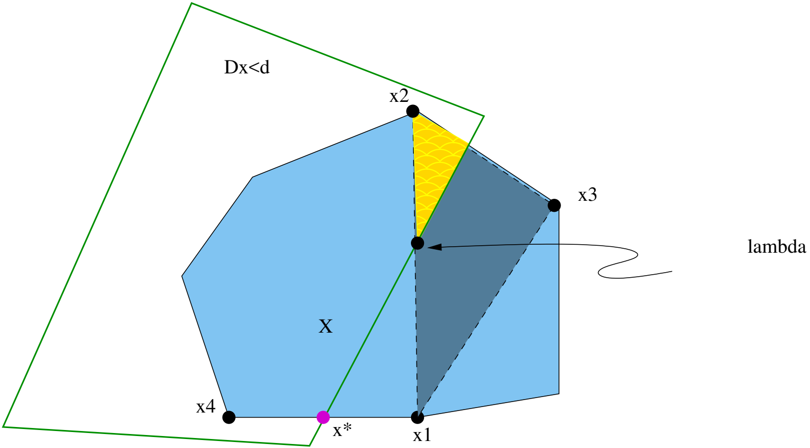
- Minimize $\mathbf{c}^\top \mathbf{x}$ when $\mathbf{x}$ lies in the convex hull of the extreme points $\mathbf{x}^i$ found so far and fulfills the constraints that are Lagrangian relaxed

An illustration in the x-space



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- Then generate an optimal dual solution $\boldsymbol{\mu}^k$ to this RMP, where the dual variables $\boldsymbol{\mu}$ correspond to the primal constraints $\boldsymbol{D}\boldsymbol{x} \leq \boldsymbol{d}$

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- Find a feasible solution to the original LP by solving the RMP
- Then generate an optimal dual solution $\boldsymbol{\mu}^k$ to this RMP, where the dual variables $\boldsymbol{\mu}$ correspond to the primal constraints $\boldsymbol{Dx} \leq \boldsymbol{d}$
- The vector $\boldsymbol{x}^{k+1}$ generated in the next subproblem (3) was already included in the RMP $\iff$
  An optimal solution to the original LP is found and verified

Three algorithms which are "dual" to each other

- "Cutting plane" applied to the Lagrangian dual of an LP

$$\Longleftrightarrow$$

- "Dantzig–Wolfe decomposition" applied to the original LP

$$\Longleftrightarrow$$

- Benders decomposition applied to the dual of the original LP

Column generation

- Consider an LP with *very* many variables: $c_j, x_j \in \mathbb{R}$, $\boldsymbol{a}_j, \boldsymbol{b} \in \mathbb{R}^m$, $n \gg m$

$$\text{minimize} \quad z = \sum_{j=1}^{n} c_j x_j \tag{6a}$$

$$\text{subject to} \quad \sum_{j=1}^{n} \boldsymbol{a}_j x_j = \boldsymbol{b} \tag{6b}$$

$$x_j \geq 0, \qquad j = 1, \ldots, n \tag{6c}$$

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- The matrix $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ is too large to handle
- Assume that m is relatively small $\Longrightarrow$ a basis matrix is not too large ($m \times m$)

Basic feasible solutions

- $B := \{m \text{ elements from the set } \{1, \ldots, n\}\}$ is a basis if the corresponding matrix $\boldsymbol{B} = (\boldsymbol{a}_j)_{j \in B}$ has an inverse, $\boldsymbol{B}^{-1}$

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    - Let $\bar{c}_s := \underset{j \notin B}{\text{minimum}} \{\bar{c}_j\}$

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  - If $\bar{c}_s < 0 \implies$ a better solution will be received if x_s enters the basis

Basic feasible solutions

- $B := \{m \text{ elements from the set } \{1, \dots, n\}\}$ is a basis if the corresponding matrix $\boldsymbol{B} = (\boldsymbol{a}_j)_{j \in B}$ has an inverse, $\boldsymbol{B}^{-1}$
- A basic solution is given by $\boldsymbol{x}_B := \boldsymbol{B}^{-1}\boldsymbol{b}$ and $x_j = 0$, $j \notin B$. It is feasible if $\boldsymbol{x}_B \geq \boldsymbol{0}^m$
- A better basic feasible solution can be found by computing the reduced costs: $\bar{c}_j := c_j - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1}\boldsymbol{a}_j$ for $j \notin B$
  - Let $\bar{c}_s := \underset{j \notin B}{\text{minimum}} \{\bar{c}_j\}$
  - If $\bar{c}_s < 0 \implies$ a better solution will be received if x_s enters the basis
  - If $\bar{c}_s \geq 0 \implies \boldsymbol{x}_B$ is an optimal basic solution

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- Suppose the columns $\boldsymbol{a}_j$ are defined by a set $S := \{\boldsymbol{a}_j \mid j = 1, \ldots, n\}$ being, e.g., solutions to a system of equations (extreme points, integer points, ...)

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- If $\bar{c}(\boldsymbol{a}') < 0$, let the column $\begin{bmatrix} c(\boldsymbol{a}') \\ \boldsymbol{a}' \end{bmatrix}$ enter the LP (6)

Example: The cutting stock problem

- **Supply:** rolls of e.g. paper of length L
- **Demand:** b_i roll pieces of length $\ell_i < L$, $i = 1, \ldots, m$
- **Objective:** minimize the number of rolls needed for producing the demanded pieces

First formulation

$$x_k = \begin{cases} 1 & \text{if roll } k \text{ is used} \\ 0 & \text{otherwise} \end{cases} \qquad y_{ik} = \begin{cases} \text{\# of pieces of length } \ell_i \\ \text{that are cut from roll } k \end{cases}$$

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{K} x_k, \\ \text{subject to} \quad & \sum_{i=1}^{m} \ell_i y_{ik} \le L x_k, \quad k = 1, \ldots, K \\ & \sum_{k=1}^{K} y_{ik} = b_i, \qquad i = 1, \ldots, m \\ & x_k \text{ binary}, \quad k = 1, \ldots, K \\ & y_{ik} \in \mathbb{Z}_+^{m \cdot K}, \; i = 1, \ldots, m, \; k = 1, \ldots, K \end{aligned}$$

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- Also, there are a lot of symmetries (i.e., equivalently good, but differently denoted, solutions) in the “first formulation”
- Symmetries are extremely difficult to handle for integer programming solvers

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$\Rightarrow$ Relax integrality constraints, solve an LP instead of an ILP

Starting solution

Natural: m unit columns (yields lots of waste) $\Longrightarrow$

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{m} x_j \\ \text{subject to} \quad & x_j = b_j, \qquad j = 1, \ldots, m \\ & x_j \geq 0, \qquad j = 1, \ldots, m \end{aligned}$$

New columns generated by the subproblem

Generate better patterns using the dual variable values $\pi_i \implies$ a new column $\overline{\boldsymbol{a}}_k$ computed from:

$$1 - \max_{a_{ik}} \left\{ \sum_{i=1}^{m} \pi_i a_{ik} \right\} \qquad \left[\Leftrightarrow \text{minimize} \left(c_k - \underbrace{\boldsymbol{c}_B^\top \boldsymbol{B}^{-1}}_{\boldsymbol{\pi}} \boldsymbol{a}_k \right) \right]$$

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- If $1 - \boldsymbol{\pi}^\top \overline{\boldsymbol{a}}_k < 0$, then the column $\overline{\boldsymbol{a}}_k$ will improve the formulation
- If $1 - \boldsymbol{\pi}^\top \overline{\boldsymbol{a}}_k \geq 0$, then the columns already generated are sufficient to determine the optimum (of the LP problem)