TMA521/MMA511 Large Scale Optimization Lecture 8 Dantzig–Wolfe decomposition, column generation, and branch–and–price

Ann-Brith Strömberg

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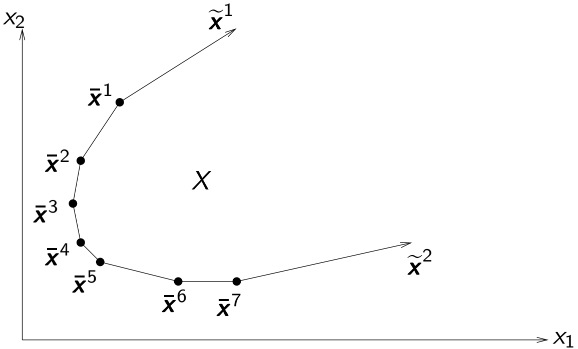
Column generation: a method for solving linear programs (LP) (and actually also MILP's) with very many columns (variables)

- The cutting stock problem—second formulation—comprises very many columns and relatively few rows
- What if not? (i.e., fairly many of both rows and columns, but "hard" due to integer requirements)
- The Dantzig-Wolfe reformulation

Formulation of a general linear program in a form suitable for column generation: Dantzig–Wolfe decomposition

- Let $X = \{\boldsymbol{x} \in \mathbb{R}^n_+ \mid \boldsymbol{Ax} = \boldsymbol{b}\}$ (or $\boldsymbol{Ax} \leq \boldsymbol{b}$) be a polyhedron with
- *extreme points* $\bar{\boldsymbol{x}}^p$, $p \in \mathcal{P}$ and
- *extreme recession directions* (extreme rays) $\tilde{\boldsymbol{x}}^r$, $r \in \mathcal{R}$



Inner representation of the set X

$$\mathbf{x} \in X \iff \left(\begin{array}{lll} \mathbf{x} & = & \displaystyle\sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p + \sum_{r \in \mathcal{R}} \mu_r \widetilde{\mathbf{x}}^r \\ & & \displaystyle\sum_{p \in \mathcal{P}} \lambda_p = 1 \\ & & \lambda_p \geq 0, \quad p \in \mathcal{P} \\ & & \mu_r \geq 0, \quad r \in \mathcal{R} \end{array} \right)$$

- $\mathbf{x} \in X$ is a *convex* combination of the extreme points plus a *conical* combination of the extreme directions
- Use this *inner representation* of the set X to reformulate a linear program according to the *Dantzig-Wolfe decomposition principle*
- Solve by *column generation*

A linear program and a corresponding master problem

$$
\begin{aligned}
\text{(LP)} \quad z^* = \text{minimum} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{D}\boldsymbol{x} = \boldsymbol{d} \quad \longleftarrow \quad \text{(complicating constraints)} \\
& \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad \longleftarrow \quad \text{(“simple” constraints)} \\
& \boldsymbol{x} \geq \mathbf{0}
\end{aligned}
$$

- Let $X = \{\, \boldsymbol{x} \geq 0 \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \,\}$
- Extreme points of X: $\bar{\boldsymbol{x}}^p$, $p \in \mathcal{P}$
- Extreme directions of X: $\widetilde{\boldsymbol{x}}^r$, $r \in \mathcal{R}$

$\Longrightarrow$

The complete master problem (MP)

$$
\text{(MP)} \quad z^* = \min_{(\lambda,\mu)} \sum_{p\in\mathcal{P}} \lambda_p(\boldsymbol{c}^\top \bar{\boldsymbol{x}}^p) + \sum_{r\in\mathcal{R}} \mu_r(\boldsymbol{c}^\top \widetilde{\boldsymbol{x}}^r)
$$

$$
\text{s.t.} \quad \sum_{p\in\mathcal{P}} \lambda_p(\boldsymbol{D}\bar{\boldsymbol{x}}^p) + \sum_{r\in\mathcal{R}} \mu_r(\boldsymbol{D}\widetilde{\boldsymbol{x}}^r) = \boldsymbol{d} \qquad | \ \boldsymbol{\pi}
$$

$$
\sum_{p\in\mathcal{P}} \lambda_p = 1 \qquad | \ q
$$

$$
\lambda_p \geq 0, \ \ p \in \mathcal{P}
$$

$$
\mu_r \geq 0, \ \ r \in \mathcal{R}
$$

- # constraints in (MP) equals # constraints in "$\boldsymbol{Dx} = \boldsymbol{d}$" + 1
- # columns very large
  (equals # extreme points & # directions of X)

The restricted master problem (RMP)

- Assume that not all extreme points/directions have been found: $\bar{\mathcal{P}} \subseteq \mathcal{P}$; $\bar{\mathcal{R}} \subseteq \mathcal{R}$

$$
\begin{aligned}
\text{(RMP)} \quad z^* \leq \min_{(\lambda,\mu)} \quad & \sum_{p \in \bar{\mathcal{P}}} \lambda_p (\boldsymbol{c}^\top \bar{\boldsymbol{x}}^p) + \sum_{r \in \bar{\mathcal{R}}} \mu_r (\boldsymbol{c}^\top \widetilde{\boldsymbol{x}}^r) \\
\text{s.t.} \quad & \sum_{p \in \bar{\mathcal{P}}} \lambda_p (\boldsymbol{D} \bar{\boldsymbol{x}}^p) + \sum_{r \in \bar{\mathcal{R}}} \mu_r (\boldsymbol{D} \widetilde{\boldsymbol{x}}^r) = \boldsymbol{d} & \quad | \; \boldsymbol{\pi} \\
& \sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 & \quad | \; q \\
& \lambda_p \geq 0, \;\; p \in \bar{\mathcal{P}} \\
& \mu_r \geq 0, \;\; r \in \bar{\mathcal{R}}
\end{aligned}
$$

- The number of constraints in (RMP) equals that of (MP)
- The number of columns is considerably smaller

The linear programming dual of the (restricted) master problem (D-RMP)

- The linear programming dual of (RMP) is given by

$$
\text{(D-RMP)} \quad z^* \leq \max_{(\boldsymbol{\pi}, q)} \ \boldsymbol{d}^\top \boldsymbol{\pi} + q
$$

$$
\begin{aligned}
\text{s.t.} \quad & (\boldsymbol{D}\bar{\boldsymbol{x}}^p)^\top \boldsymbol{\pi} + q \leq (\boldsymbol{c}^\top \bar{\boldsymbol{x}}^p), \quad p \in \bar{\mathcal{P}} \quad & | \ \lambda_p \\
& (\boldsymbol{D}\widetilde{\boldsymbol{x}}^r)^\top \boldsymbol{\pi} \qquad \leq (\boldsymbol{c}^\top \widetilde{\boldsymbol{x}}^r), \quad r \in \bar{\mathcal{R}} \quad & | \ \mu_r
\end{aligned}
$$

  with solution $(\bar{\boldsymbol{\pi}}, \bar{q})$

- Reduced cost for the variable λ_p, $p \in \mathcal{P} \setminus \bar{\mathcal{P}}$ $(\Leftrightarrow \bar{\boldsymbol{x}}^p)$:
  $$(\boldsymbol{c}^\top \bar{\boldsymbol{x}}^p) - (\boldsymbol{D}\bar{\boldsymbol{x}}^p)^\top \bar{\boldsymbol{\pi}} - \bar{q} = (\boldsymbol{c} - \boldsymbol{D}^\top \bar{\boldsymbol{\pi}})^\top \bar{\boldsymbol{x}}^p - \bar{q}$$

- Reduced cost for the variable μ_r, $r \in \mathcal{R} \setminus \bar{\mathcal{R}}$ $(\Leftrightarrow \tilde{\boldsymbol{x}}^r)$:
  $$(\boldsymbol{c}^\top \widetilde{\boldsymbol{x}}^r) - (\boldsymbol{D}\widetilde{\boldsymbol{x}}^r)^\top \bar{\boldsymbol{\pi}} = (\boldsymbol{c} - \boldsymbol{D}^\top \bar{\boldsymbol{\pi}})^\top \widetilde{\boldsymbol{x}}^r$$

Column generation

- The smallest reduced cost is found by solving the column generation subproblem to

$$\min_{\mathbf{x}\in X} (\boldsymbol{c} - \boldsymbol{D}^\top \bar{\boldsymbol{\pi}})^\top \boldsymbol{x} \quad \left(\text{alt:} \quad \min_{\mathbf{x}\in X} (\boldsymbol{c} - \boldsymbol{D}^\top \bar{\boldsymbol{\pi}})^\top \boldsymbol{x} - \bar{q}\right)$$

$$\Longleftrightarrow \min \left\{ \min_{p\in\mathcal{P}} \left\{ (\boldsymbol{c} - \boldsymbol{D}^\top \bar{\boldsymbol{\pi}})^\top \bar{\boldsymbol{x}}^p \right\} - \bar{q} \; ; \; \min_{r\in\mathcal{R}} \left\{ (\boldsymbol{c} - \boldsymbol{D}^\top \bar{\boldsymbol{\pi}})^\top \tilde{\boldsymbol{x}}^r \right\} \right\}$$

- Gives as solution an extreme point, $\bar{\boldsymbol{x}}^p$, or an extreme direction $\widetilde{\boldsymbol{x}}^r$
- Unbounded solutions can be detected within the simplex method! How?

$\Longrightarrow$ a new column in (RMP) (if the reduced cost < 0):

- Either $\begin{pmatrix} \boldsymbol{c}^\top \bar{\boldsymbol{x}}^p \\ \boldsymbol{D}\bar{\boldsymbol{x}}^p \\ 1 \end{pmatrix}$ or $\begin{pmatrix} \boldsymbol{c}^\top \widetilde{\boldsymbol{x}}^r \\ \boldsymbol{D}\widetilde{\boldsymbol{x}}^r \\ 0 \end{pmatrix}$ enters the problem and improves the solution

A small integer linear optimization example of Dantzig-Wolfe decomposition and column generation

(ILP)

$$\begin{aligned} z^*_{\text{ILP}} \quad = \min \;& 2x_1 + 3x_2 + \;\; x_3 + 4x_4 \\ \text{s.t. } & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 \qquad\qquad \mid \boldsymbol{Dx} = \boldsymbol{d} \\ & \;\; x_1 + \;\; x_2 + \;\; x_3 + \;\; x_4 = 2 \\ & \;\; x_1 \;, \;\; x_2 \;, \;\; x_3 \;, \;\; x_4 \in \{0, 1\} \end{aligned}$$

- $X_{\text{ILP}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$
  $= \{\bar{\boldsymbol{x}}^1, \dots, \bar{\boldsymbol{x}}^6\}$
- Optimal solution: $\boldsymbol{x}^*_{\text{ILP}} = (0, 1, 1, 0)^\top$
- Optimal value: $z^*_{\text{ILP}} = 4$

Linear programming relaxation

(LP)

$$
\begin{aligned}
z^* = \min \ & 2x_1 + 3x_2 + x_3 + 4x_4 && [\boldsymbol{c}^\top \boldsymbol{x}] \\
\text{s.t.} \ & 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 && [\boldsymbol{D}\boldsymbol{x} = \boldsymbol{d}] \\
& x_1 + x_2 + x_3 + x_4 = 2 && [\boldsymbol{x} \in X] \\
0 \leq \ & x_1 \ , \ x_2 \ , \ x_3 \ , \ x_4 \leq 1 && [\boldsymbol{x} \in X]
\end{aligned}
$$

- $X = \operatorname{conv} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

  $= \operatorname{conv} \{ \bar{\boldsymbol{x}}^1, \ldots, \bar{\boldsymbol{x}}^6 \}$

  $= \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \boldsymbol{x} = \sum_{p=1}^{6} \lambda_p \bar{\boldsymbol{x}}^p ; \sum_{p=1}^{6} \lambda_p = 1 ; \lambda_p \geq 0, p = 1, \ldots, 6 \right\}$

The complete master problem and the initial columns

(MP)

$$\begin{aligned} z^* = \min\ & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 + 7\lambda_5 + 5\lambda_6 \\ \text{s.t.}\ & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 + 4\lambda_5 + 5\lambda_6 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 \\ & \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6 \geq 0 \end{aligned}$$

- Initial columns: $\lambda_1, \lambda_2, \lambda_3$

(RMP)

$$\begin{aligned} z^* \leq \min\ & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 \\ \text{s.t.}\ & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1,\ \lambda_2,\ \lambda_3 \geq 0 \end{aligned}$$

(D-RMP)

$$\begin{aligned} z^* \leq \max\ & 5\pi + q \\ \text{s.t.}\ & 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \end{aligned}$$

- Solution: $\bar{\boldsymbol{\lambda}} = (1, 0, 0)^\top$, $\bar{\pi} = -2, \quad \bar{q} = 15$

Reduced costs computation

$$\min_{\mathbf{x}\in X} \left\{ (\boldsymbol{c} - \boldsymbol{D}^\top \bar{\pi})^\top \boldsymbol{x} - \bar{q} \right\} = \min_{p=1,\ldots,6} \left\{ (\boldsymbol{c} - \boldsymbol{D}^\top \bar{\pi})^\top \bar{\boldsymbol{x}}^p - \bar{q} \right\}$$
$$= \min_{p=1,\ldots,6} \left\{ [(2,3,1,4) - (3,2,3,2) \cdot (-2)]\, \bar{\boldsymbol{x}}^p - 15 \right\}$$
$$= \min \{0,0,1,-1,0,0\} = -1 < 0$$

- New extreme point found in (LP): $\bar{\boldsymbol{x}}^4 = (0,1,1,0)^\top$

$\Rightarrow$ New column in (RMP): $\begin{pmatrix} \boldsymbol{c}^\top \bar{\boldsymbol{x}}^4 \\ \boldsymbol{D}\bar{\boldsymbol{x}}^4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

New, extended restricted master problem

(RMP)

$$
\begin{aligned}
z^* \le \min\ & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 \\
\text{s.t. } & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 = 5 \\
& \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\
& \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4 \ge 0
\end{aligned}
$$

(D-RMP)

$$
\begin{aligned}
z^* \le \max\ & 5\pi + q \\
\text{s.t. } & 5\pi + q \le 5 \\
& 6\pi + q \le 3 \\
& 5\pi + q \le 6 \\
& 5\pi + q \le 4
\end{aligned}
$$

- Solution: $\bar{\boldsymbol{\lambda}} = (0,0,0,1)^\top$, $\bar{\pi} = -1$, $\bar{q} = 9$
- Reduced costs:
  $\min_{p=1,\dots,6} \left\{ (5,5,4,6)\,\bar{\mathbf{x}}^p - 9 \right\} = \min\{1,0,2,0,2,1\} = 0$

Optimal solution to (MP) and to (LP)

- $\boldsymbol{\lambda}^* = (0,0,0,1,0,0)^\top, \qquad \pi^* = -1, \quad q^* = 9$

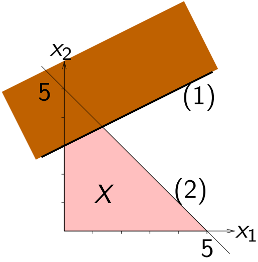
$\Longrightarrow$ $\boldsymbol{x}^* = \bar{\boldsymbol{x}}^4 = (0,1,1,0)^\top = \boldsymbol{x}^*_{\text{ILP}}, \qquad z^* = 4 = z^*_{\text{ILP}}$

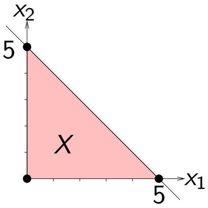
- A coincidence that the solution was integral!
- In general, the solution $\boldsymbol{x}^*$ to (LP) may have fractional variable values
- Solution to (ILP)?
- Need to find an integer solution (not certainly an optimal solution to (ILP), why?) among the columns generated, i.e., solve

$$\min \left\{ (2,3,1,4)\boldsymbol{x} \,\middle|\, (3,2,3,2)\boldsymbol{x} = 5, \;\; \boldsymbol{x} \in \left\{ \bar{\boldsymbol{x}}^1, \bar{\boldsymbol{x}}^2, \bar{\boldsymbol{x}}^3, \bar{\boldsymbol{x}}^4 \right\} \right\}$$

Another numerical example of Dantzig-Wolfe decomposition and column generation

$$
\begin{array}{lrcrcl}
\min & x_1 & - & 3x_2 & & & (0) \\
\text{st} & -x_1 & + & 2x_2 & \leq & 6 & (1) \quad \longleftarrow \quad \text{(complicating)} \\
& x_1 & + & x_2 & \leq & 5 & (2) \\
& x_1 & , & x_2 & \geq & 0 & (3)
\end{array}
$$





$$X = \left\{ \boldsymbol{x} \in \mathbb{R}^2_+ \,\middle|\, x_1 + x_2 \leq 5 \right\} = \operatorname{conv} \left\{ (0,0)^\top, (0,5)^\top, (5,0)^\top \right\}$$

Complete Dantzig-Wolfe master problem

$$\boldsymbol{x} \in X \iff \left\{ \begin{array}{r} \boldsymbol{x} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\lambda_3 \\ 5\lambda_2 \end{pmatrix} \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \right\}$$

$$\begin{array}{llr} \min & -15\lambda_2 + 5\lambda_3 & (0) \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 & (1) \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 & \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 & \end{array}$$

- The first (RMP) is then constructed from the points $(0, 0)^\top$ and $(0, 5)^\top$ (corresponds to λ_1 and λ_2)

Iteration 1

- $$\begin{array}{rrcl} \min & - & 15\lambda_2 & (0) \\ \text{s.t.} & & 10\lambda_2 \leq 6 & (1) \\ & \lambda_1 + \lambda_2 & = 1 & \\ & \lambda_1, \lambda_2 & \geq 0 & \end{array} \quad \left| \begin{array}{ll} \text{Solution:} & \bar{\boldsymbol{\lambda}} = (\frac{2}{5}, \frac{3}{5})^\top \\ \text{Dual solution:} & \bar{\pi} = -\frac{3}{2}, \bar{q} = 0 \end{array} \right.$$

- Smallest reduced cost:

$$\min_{\mathbf{x} \in X} \left[(\boldsymbol{c}^\top - \bar{\pi} \boldsymbol{D}) \boldsymbol{x} - \bar{q} \right] = \min_{\mathbf{x} \in X} \left(\left[(1, -3) - (-\tfrac{3}{2})(-1, 2) \right] \boldsymbol{x} - 0 \right)$$

$$= \min \left\{ -\tfrac{1}{2} x_1 \mid x_1 + x_2 \leq 5; \boldsymbol{x} \geq \mathbf{0}^2 \right\} = -\tfrac{5}{2} < 0 \Longrightarrow \bar{\boldsymbol{x}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

- New column:

$$\left. \begin{array}{r} \boldsymbol{c}^\top \bar{\boldsymbol{x}} = (1, -3)(5, 0)^\top = 5 \\ \boldsymbol{D} \bar{\boldsymbol{x}} = (-1, 2)(5, 0)^\top = -5 \end{array} \right\} \Longrightarrow \begin{pmatrix} \frac{5}{-5} \\ 1 \end{pmatrix}$$

Iteration 2

$$\begin{array}{rl} \min & -15\lambda_2 + 5\lambda_3 \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array}$$

Solution:	$\bar{\boldsymbol{\lambda}} = (0, \frac{11}{15}, \frac{4}{15})^\top$
Dual solution:	$\bar{\pi} = -\frac{4}{3}, \bar{q} = -\frac{5}{3}$

- Smallest reduced cost:

$$\min_{\mathbf{x} \in X} \left[(\boldsymbol{c}^\top - \bar{\pi} \boldsymbol{D}) \boldsymbol{x} - \bar{q} \right] = \min_{\mathbf{x} \in X} \left(\left[(1, -3) - (-\tfrac{4}{3})(-1, 2) \right] \boldsymbol{x} - (-\tfrac{5}{3}) \right)$$

$$= \min \left\{ -\tfrac{1}{3} x_1 - \tfrac{1}{3} x_2 + \tfrac{5}{3} \mid x_1 + x_2 \leq 5; \boldsymbol{x} \geq \mathbf{0}^2 \right\} = 0$$

- Optimal solution: $\boldsymbol{\lambda}^* = \left(0, \dfrac{11}{15}, \dfrac{4}{15} \right)^\top$

$\Longrightarrow \boldsymbol{x}^* = (5\lambda_3, 5\lambda_2)^\top = (\frac{4}{3}, \frac{11}{3})^\top; \quad z^* = \frac{4}{3} - 3 \cdot \frac{11}{3} = -9\frac{2}{3}$

Upper bound on the optimal objective value for (LP)

- The complete master problem (MP):

$$
\begin{aligned}
z^* = \min \ & \sum_{p \in \mathcal{P}} \lambda_p (\boldsymbol{c}^\top \bar{\boldsymbol{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\boldsymbol{c}^\top \widetilde{\boldsymbol{x}}^r) \\
\text{s.t.} \ & \sum_{p \in \mathcal{P}} \lambda_p (\boldsymbol{D}\bar{\boldsymbol{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\boldsymbol{D}\widetilde{\boldsymbol{x}}^r) = \boldsymbol{d} \qquad | \ \boldsymbol{\pi} \\
& \sum_{p \in \mathcal{P}} \lambda_p = 1 \qquad | \ q \\
& \lambda_p, \mu_r \geq 0, \ p \in \mathcal{P}, r \in \mathcal{R}
\end{aligned}
$$

- The (dual of the) restricted master problem (D-RMP) yields an upper bound on z^*:

$$
\begin{aligned}
z^* \leq \bar{z} = \boldsymbol{d}^\top \bar{\boldsymbol{\pi}} + \bar{q} =: \max_{(\boldsymbol{\pi}, q)} \ & \boldsymbol{d}^\top \boldsymbol{\pi} + q \\
\text{s.t.} \ & (\boldsymbol{D}\bar{\boldsymbol{x}}^p)^\top \boldsymbol{\pi} + q \leq \boldsymbol{c}^\top \bar{\boldsymbol{x}}^p, \quad p \in \bar{\mathcal{P}} \\
& (\boldsymbol{D}\widetilde{\boldsymbol{x}}^r)^\top \boldsymbol{\pi} \leq \boldsymbol{c}^\top \widetilde{\boldsymbol{x}}^r, \quad r \in \bar{\mathcal{R}}
\end{aligned}
$$

Lower bound on the optimal objective value for (LP)

- Let λ_p^*, $p \in \mathcal{P}$, and μ_r^*, $r \in \mathcal{R}$, be optimal in the complete master problem (MP)
- Let $(\bar{\boldsymbol{\pi}}, \bar{q})$ be an optimal dual solution for the restricted master problem (RMP), with columns corresponding to $\bar{\mathcal{P}}$ and $\bar{\mathcal{R}}$
- Multiply the right-hand side elements of the primal (i.e., $\boldsymbol{d}$ and 1) by $\bar{\boldsymbol{\pi}}$ and $\bar{q}$, respectively

$\Longrightarrow$ (1'st ineq. from dual, 2'nd ineq. since $\sum \lambda_p^* = 1$)

$$
\begin{aligned}
0 \geq\; & z^* - \bar{z} = z^* - \boldsymbol{d}^\top \bar{\boldsymbol{\pi}} - 1 \cdot \bar{q} \\
=\; & \sum_{p \in \mathcal{P}} \lambda_p^* \left[\boldsymbol{c}^\top \bar{\boldsymbol{x}}^p - (\boldsymbol{D}\bar{\boldsymbol{x}}^p)^\top \bar{\boldsymbol{\pi}} - \bar{q} \right] + \sum_{r \in \mathcal{R}} \mu_r^* \left[\boldsymbol{c}^\top \tilde{\boldsymbol{x}}^r - (\boldsymbol{D}\tilde{\boldsymbol{x}}^r)^\top \bar{\boldsymbol{\pi}} \right] \\
\geq\; & \min_{p \in \mathcal{P}} \left[\boldsymbol{c}^\top \bar{\boldsymbol{x}}^p - (\boldsymbol{D}\bar{\boldsymbol{x}}^p)^\top \bar{\boldsymbol{\pi}} - \bar{q} \right] + \sum_{r \in \mathcal{R}} \mu_r^* \min_{s \in \mathcal{R}} \left[\boldsymbol{c}^\top \tilde{\boldsymbol{x}}^s - (\boldsymbol{D}\tilde{\boldsymbol{x}}^s)^\top \bar{\boldsymbol{\pi}} \right]
\end{aligned}
$$

Lower bound on the optimal objective value for (LP)

- If the subproblem has an unbounded solution then no optimistic estimate (i.e., lower bound since we minimize) can be computed in this iteration
- Otherwise it holds that
$$\min_{s \in \mathcal{R}} \left[\boldsymbol{c}^\top \widetilde{\boldsymbol{x}}^s - (\boldsymbol{D}\widetilde{\boldsymbol{x}}^s)^\top \bar{\boldsymbol{\pi}} \right] \geq 0$$

$\Longrightarrow$

$$\begin{aligned} \bar{z} \geq z^* \geq \bar{z} &+ \min_{p \in \mathcal{P}} \left[(\boldsymbol{c} - \boldsymbol{D}^\top \bar{\boldsymbol{\pi}})^\top \bar{\boldsymbol{x}}^p - \bar{q} \right] \\ &= \bar{z} + \min_{\mathbf{x} \in X} (\boldsymbol{c} - \boldsymbol{D}^\top \bar{\boldsymbol{\pi}})^\top \boldsymbol{x} - \bar{q} \\ &=: \underline{z} \end{aligned}$$

Convergence of the column generation algorithm

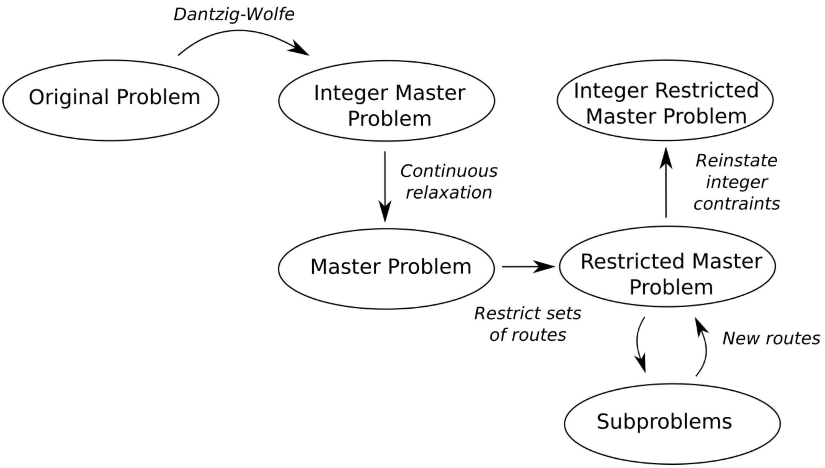
- The number of columns generated is finite, since X is polyhedral
- When no more columns are generated, the solution to the last restricted master problem will also solve (LP)
- For each new column that is added to the restricted master problem (RMP), its optimal objective value will decrease

$\Rightarrow$ The pessimistic estimate $\bar{z}_k$ converges monotonically to z^*

- The optimistic estimate $\underline{z}_k$ also converges, but not monotonically
- If at iteration k an optimal solution to the complete master problem (MP) is received, then $\underline{z}_k = \bar{z}_k$ holds

$\Rightarrow$ Stopping criterion: $\bar{z}_k - \underline{z}_k^* \leq \varepsilon$, where $\underline{z}_k^* = \max_{s=1,\ldots,k} \underline{z}_s$ and $\varepsilon > 0$

Column generation



Dantzig-Wolfe decomposition applied to a linear program with block-angular structure

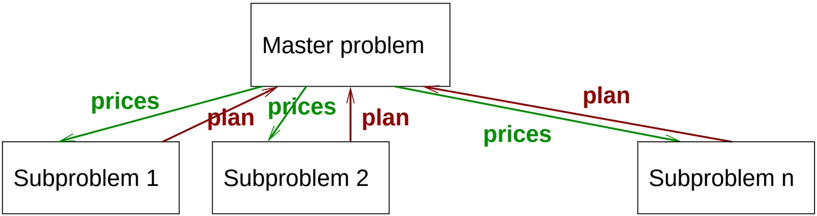
$$
\begin{array}{llrl}
\text{(LP)} & \min & \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 + \cdots + \mathbf{c}_n^\top \mathbf{x}_n & \\
 & \text{s.t.} & \mathbf{D}_1 \mathbf{x}_1 + \mathbf{D}_2 \mathbf{x}_2 + \cdots + \mathbf{D}_n \mathbf{x}_n \leq \mathbf{d} & | \quad \text{Dual var: } \boldsymbol{\pi} \\
 & & \mathbf{A}_1 \mathbf{x}_1 \qquad\qquad\qquad\qquad \leq \mathbf{b}_1 & | \quad \mathbf{x}_1 \in X_1 \\
 & & \mathbf{A}_2 \mathbf{x}_2 \qquad\qquad \leq \mathbf{b}_2 & | \quad \mathbf{x}_2 \in X_2 \\
 & & \cdots \qquad\quad \cdot\ \cdot & \\
 & & \mathbf{A}_n \mathbf{x}_n \leq \mathbf{b}_n & | \quad \mathbf{x}_n \in X_n \\
 & & \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \geq \mathbf{0} &
\end{array}
$$

Cartesian product set:

$$X = X_1 \times X_2 \times \ldots \times X_n$$

Dantzig-Wolfe decomposition interpreted as decentralized planning

- The main office (master problem) sets prizes ($\boldsymbol{\pi}$) for the common resources (complicating constraints)
- The departments (subproblems) suggest (production) plans (i.e., columns) ($\boldsymbol{D}_j \bar{\boldsymbol{x}}_j^p$) based on given prices
- The main office "mixes" the suggested plans (columns) optimally; sets new prices
- The procedure is repeated



Inner representations of the sets X_j, $j = 1, \ldots, n$

- Let $X_j = \{\boldsymbol{x}_j \geq \boldsymbol{0} \mid \boldsymbol{A}_j \boldsymbol{x}_j \leq \boldsymbol{b}_j\}$ and express $\boldsymbol{x}_j$ as

$$\boldsymbol{x}_j \in X_j \iff \left(\begin{array}{rcl} \boldsymbol{x}_j & = & \displaystyle\sum_{p \in \mathcal{P}_j} \lambda_{pj} \bar{\boldsymbol{x}}_j^p + \sum_{r \in \mathcal{R}_j} \mu_{rj} \widetilde{\boldsymbol{x}}_j^r \\ & & \displaystyle\sum_{p \in \mathcal{P}_j} \lambda_{pj} = 1 \\ & & \lambda_{pj} \geq 0, \quad p \in \mathcal{P}_j \\ & & \mu_{rj} \geq 0, \quad r \in \mathcal{R}_j \end{array} \right) \quad j = 1, \ldots, n$$

$$\iff$$

- $X_j = \texttt{conv}\left\{\bar{\boldsymbol{x}}_j^p \mid p \in \mathcal{P}_j\right\} + \texttt{cone}\left\{\tilde{\boldsymbol{x}}_j^r \mid r \in \mathcal{R}_j\right\}, \quad j = 1, \ldots, n$

The complete master problem (MP)

$$
\text{(MP)} \quad \min_{(\lambda,\mu)} \sum_{j=1}^{n} \left(\sum_{p \in \mathcal{P}_j} \lambda_{pj} (\boldsymbol{c}^\top \bar{\boldsymbol{x}}_j^p) + \sum_{r \in \mathcal{R}_j} \mu_{rj} (\boldsymbol{c}^\top \tilde{\boldsymbol{x}}_j^r) \right)
$$

$$
\text{s.t.} \quad \sum_{j=1}^{n} \left(\sum_{p \in \mathcal{P}_j} \lambda_{pj} (\boldsymbol{D}\bar{\boldsymbol{x}}_j^p) + \sum_{r \in \mathcal{R}_j} \mu_{rj} (\boldsymbol{D}\tilde{\boldsymbol{x}}_j^r) \right) \leq \boldsymbol{d}
$$

$$
\sum_{p \in \mathcal{P}_j} \lambda_{pj} = 1, \qquad j = 1, \dots, n
$$

$$
\lambda_{pj} \geq 0, \;\; p \in \mathcal{P}_j, \;\; j = 1, \dots, n
$$

$$
\mu_{rj} \geq 0, \;\; r \in \mathcal{R}_j, \;\; j = 1, \dots, n
$$

- The number of constraints in the master problem equals the number of constraints in "$\boldsymbol{Dx} \leq \boldsymbol{d}$" $+ \; n$
- The restricted master problem (RMP) is formulated analogously as before, with $\bar{\mathcal{P}}_j \subseteq \mathcal{P}_j$ and $\bar{\mathcal{R}}_j \subseteq \mathcal{R}_j$, $j = 1, \dots, n$

Use the dual solution to generate new columns

- The linear programming dual of the restricted master problem

$$
\text{(D-RMP)} \quad \max_{(\boldsymbol{\pi},\mathbf{q})} \quad \boldsymbol{d}^\top \boldsymbol{\pi} + \sum_{j=1}^n q_j
$$

$$
\begin{aligned}
\text{s.t.} \quad & (\boldsymbol{D}_j \bar{\boldsymbol{x}}_j^p)^\top \boldsymbol{\pi} + q_j \geq (\boldsymbol{c}_j^\top \bar{\boldsymbol{x}}_j^p), && p \in \bar{\mathcal{P}}_j,\ j = 1, \ldots, n, && \mid \lambda_{pj} \\
& (\boldsymbol{D}_j \widetilde{\boldsymbol{x}}_j^r)^\top \boldsymbol{\pi} \geq (\boldsymbol{c}_j^\top \widetilde{\boldsymbol{x}}_j^r), && r \in \bar{\mathcal{R}}_j,\ j = 1, \ldots, n, && \mid \mu_{rj} \\
& \boldsymbol{\pi} \leq \mathbf{0},
\end{aligned}
$$

with solution $\bar{\boldsymbol{\pi}}$, $\bar{q}_j$, $j = 1, \ldots, n$

- Generate new columns (minimization $\Leftrightarrow$ reduced cost < 0):

For $j = 1, \ldots, n$, solve $\boxed{\min_{\boldsymbol{x}_j \in X_j} \left\{ \left(\boldsymbol{c}_j - \boldsymbol{D}_j^\top \bar{\boldsymbol{\pi}} \right)^\top \boldsymbol{x}_j \right\} - \bar{q}_j}$

$\Rightarrow$ Subproblem solutions: $\bar{\boldsymbol{x}}_j^p$ or $\widetilde{\boldsymbol{x}}_j^r$

$\Rightarrow$ New columns: $\begin{pmatrix} \boldsymbol{c}_j^\top \bar{\boldsymbol{x}}_j^p \\ \hline \boldsymbol{D}_j^\top \bar{\boldsymbol{x}}_j^p \\ \boldsymbol{e}_j \end{pmatrix}$ or $\begin{pmatrix} \boldsymbol{c}_j^\top \widetilde{\boldsymbol{x}}_j^r \\ \hline \boldsymbol{D}_j^\top \widetilde{\boldsymbol{x}}_j^r \\ \mathbf{0} \end{pmatrix}$ for $j = 1, \ldots, n$, where $\boldsymbol{e}_j$ denotes the jth unit column

Find feasible solutions (right-hand side allocation)

- If the (RMP) is not solved to optimality, let $\bar{\lambda}_{pj}$, $p \in \bar{\mathcal{P}}_j$, and $\bar{\mu}_{rj}$, $r \in \bar{\mathcal{R}}_j$, $j = 1, \dots, n$, denote a feasible (and almost optimal) solution to (RMP). It then holds that

$$\sum_{j=1}^{n} \boldsymbol{D}_j \underbrace{\left(\sum_{p \in \bar{\mathcal{P}}_j} \bar{\lambda}_{pj} \bar{\boldsymbol{x}}_j^p + \sum_{r \in \bar{\mathcal{R}}_j} \bar{\mu}_{rj} \widetilde{\boldsymbol{x}}_j^r \right)}_{\in X_j} \leq \boldsymbol{d} \qquad (*)$$

- If the (RMP) solution is close to optimal, a good feasible solution to (LP) is given by a solution, for $j = 1, \dots, n$, to

$$\begin{aligned} \min \ & \boldsymbol{c}_j^\top \boldsymbol{x}_j \\ \text{s.t.} \ & \boldsymbol{D}_j \boldsymbol{x}_j \leq \sum_{p \in \bar{\mathcal{P}}_j} \bar{\lambda}_{pj} (\boldsymbol{D}_j \bar{\boldsymbol{x}}_j^p) + \sum_{r \in \bar{\mathcal{R}}_j} \bar{\mu}_{rj} (\boldsymbol{D}_j \widetilde{\boldsymbol{x}}_j^r) \qquad (**) \\ & \boldsymbol{x}_j \in X_j \end{aligned}$$

where $(*)$ and $(**)$ $\Longrightarrow \sum_{j=1}^{n} \boldsymbol{D}_j \boldsymbol{x}_j \leq \boldsymbol{d}$ holds

Branch–and–price for integer linear optimization problems

Branch–and–price for integer linear optimization problems

For ease of notation, we here consider binary variables

$$\text{(ILP)} \quad z^*_{\text{ILP}} = \min \ \boldsymbol{c}^\top \boldsymbol{x}$$
$$\text{s.t.} \ \boldsymbol{D}\boldsymbol{x} = \boldsymbol{d}$$
$$\boldsymbol{x} \in X = \{\boldsymbol{x} \in \mathbb{B}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\} = \{\bar{\boldsymbol{x}}^p \mid p \in \mathcal{P}\}$$

- Inner representation (and convexification):

$$\operatorname{conv} X = \left\{ \boldsymbol{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\boldsymbol{x}}^p \;\middle|\; \sum_{p \in \mathcal{P}} \lambda_p = 1; \ \lambda_p \geq 0, \ p \in \mathcal{P} \right\}$$

- Define $c_p := \boldsymbol{c}^\top \bar{\boldsymbol{x}}^p$ and $\boldsymbol{d}_p := \boldsymbol{D}\bar{\boldsymbol{x}}^p$, $p \in \mathcal{P}$

A stronger formulation: The complete integer master problem (IMP)

$$
\text{(IMP)} \qquad z^*_{\text{ILP}} = z^*_{\text{IMP}} := \min \sum_{p \in \mathcal{P}} c_p \lambda_p
$$

$$
\text{s.t.} \sum_{p \in \mathcal{P}} \boldsymbol{d}_p \lambda_p = \boldsymbol{d}
$$

$$
\sum_{p \in \mathcal{P}} \lambda_p = 1
$$

$$
\lambda_p \in \{0, 1\}, \qquad p \in \mathcal{P}
$$

The complete master problem (MP)

- The continuous relaxation of (IMP) to (MP) (i.e., relax $\lambda_p \in \{0,1\}$ to $\lambda_p \geq 0$) yields the same lower bound as the Lagrangian dual w.r.t. the constraints $\boldsymbol{Dx} = \boldsymbol{d}$ (i.e., z_{L}^*)

$$
\text{(MP)} \qquad z_{\mathrm{IMP}}^* \geq z_{\mathrm{MP}}^* := \min \sum_{p \in \mathcal{P}} c_p \lambda_p
$$

$$
\text{s.t.} \sum_{p \in \mathcal{P}} \boldsymbol{d}_p \lambda_p = \boldsymbol{d}
$$

$$
\sum_{p \in \mathcal{P}} \lambda_p = 1
$$

$$
\lambda_p \geq 0, \qquad p \in \mathcal{P}
$$

- The continuous relaxation (LP) of (ILP) is never better than any Lagrangian dual bound

$\Rightarrow$ $z_{\mathrm{LP}}^* \leq z_{\mathrm{L}}^* = z_{\mathrm{MP}}^* \leq z_{\mathrm{IMP}}^* = z_{\mathrm{ILP}}^*$

Restricted (continuous) master problem (RMP)

- Let $\bar{\mathcal{P}} \subseteq \mathcal{P}$.
  (RMP) is a restriction of (MP) $\iff$ (MP) is a relaxation of (RMP)

$$
\begin{aligned}
\text{(RMP)} \quad z^*_{\text{IMP}} \geq z^*_{\text{MP}} \leq \bar{z}_{\text{RMP}} := \min \ & \sum_{p \in \bar{\mathcal{P}}} c_p \lambda_p \\
\text{s.t.} \ & \sum_{p \in \bar{\mathcal{P}}} \boldsymbol{d}_p \lambda_p = \boldsymbol{d} \\
& \sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 \qquad (*) \\
& \lambda_p \geq 0, \qquad p \in \bar{\mathcal{P}}
\end{aligned}
$$

- Generate columns $\begin{pmatrix} c_p \\ \boldsymbol{d}_p \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{c}^\top \bar{\boldsymbol{x}}^p \\ \boldsymbol{D}\bar{\boldsymbol{x}}^p \\ 1 \end{pmatrix}$ until an (almost) optimal solution to (MP), $\widehat{\lambda}_p$, $p \in \bar{\mathcal{P}}$, is found
- The corresponding solution to (LP): $\widehat{\boldsymbol{x}} = \sum_{p \in \bar{\mathcal{P}}} \widehat{\lambda}_p \bar{\boldsymbol{x}}^p$

Branching over a variable x_j with $0 < \hat{x}_j < 1$

[left branch] $x_j = 0$ or $x_j = 1$ [right branch]

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 0 \qquad x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 1$$

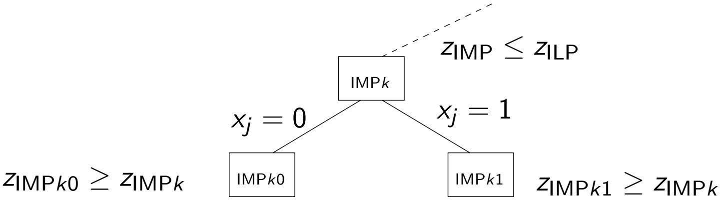
$$\Downarrow \qquad\qquad \Downarrow$$

[delete col's] $\displaystyle\sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 1} \lambda_p = 0$ $\qquad$ $\displaystyle\sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 1} \lambda_p = 1$ [replaces $(*)$]

$$\Updownarrow \qquad\qquad \Updownarrow$$

[replaces $(*)$] $\displaystyle\sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 0} \lambda_p = 1$ $\qquad$ $\displaystyle\sum_{p \in \bar{\mathcal{P}} : \bar{x}_j^p = 0} \lambda_p = 0$ [delete col's]

Generate columns in B&B nodes



- In each node (IMP), (IMP0), (IMP1), ...): Generate columns until it is (almost) optimal (i.e., all reduced costs ≥ 0) or it is verified infeasible
- If $\boldsymbol{x}^*_{\text{IMP}k\ell\ldots}$ feasible $\implies z^*_{\text{IMP}k\ell\ldots} \geq z^*_{\text{ILP}} \implies$ Cut off the branch $(k, \ell, \ldots)$

$\implies$ Cut branches $(r, s, \ldots)$ with $z^*_{IMPrs\ldots} \geq z^*_{IMPk\ell\ldots}$

The column generation subproblem, reduced costs

- $\min_{\mathbf{x} \in X^k} (\boldsymbol{c} - \boldsymbol{D}^\top \widehat{\boldsymbol{\pi}}^k)^\top \boldsymbol{x} - \widehat{q}^k =: (\boldsymbol{c} - \boldsymbol{D}^\top \widehat{\boldsymbol{\pi}}^k)^\top \bar{\boldsymbol{x}}^p - \widehat{q}^k =: \bar{c}(\bar{\boldsymbol{x}}^p)$
- $(\widehat{\boldsymbol{\pi}}^k, \widehat{q}^k)$ is a dual solution to the (RMP) and $X^k = X \cap \{\boldsymbol{x} \mid x_j = k\}$, $k \in \{0, 1\}$ (etc. down the tree)
- If $\bar{c}(\bar{\boldsymbol{x}}^p) < 0$ then $\begin{pmatrix} \boldsymbol{c}^\top \bar{\boldsymbol{x}}^p \\ \boldsymbol{D}\bar{\boldsymbol{x}}^p \\ 1 \end{pmatrix}$ is a new column in (IMPk)
- Minimization of the subproblem? A solution (column) $\bar{\boldsymbol{x}}^r$ is good enough if $\bar{c}(\bar{\boldsymbol{x}}^r) < 0$
- If $\bar{c}(\bar{\boldsymbol{x}}^p) \geq 0$ then no more columns are needed to solve (IMPk) to optimality
- In general, several columns can be generated simultaneously, i.e., save all columns r with $\bar{c}(\bar{\boldsymbol{x}}^r) < 0$ (not only the optimal)
- The same columns may be generated in different nodes $\Longrightarrow$ create "column pool" to check w.r.t. reduced costs $\bar{c}$

An instance solved by Branch–and–price

$$z^*_{\text{ILP}} = \min \quad x_1 + 2x_2 \quad = z^*_{\text{IMP}} \quad \geq z^*_{\text{MP}} \geq z^*_{LP} = \min \quad x_1 + 2x_2$$

$$\text{s.t.} \quad 2x_1 + 2x_2 \geq 1 \qquad\qquad \text{s.t.} \quad 2x_1 + 2x_2 \geq 1$$

$$x_1, x_2 \in \{0, 1\} \qquad\qquad 0 \leq x_1, x_2 \leq 1$$

$$\text{conv}X = \text{conv}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \lambda_3 + \lambda_4 \\ \lambda_2 + \lambda_4 \end{pmatrix} \middle| \sum_{p=1}^{4} \lambda_p = 1; \begin{matrix} \lambda_p \geq 0 \\ \forall p \end{matrix} \right\}$$

(MP)

$$\begin{aligned} z^*_{\text{MP}} = \min \quad & 2\lambda_2 + \lambda_3 + 3\lambda_4 \\ \text{s.t.} \quad & 2\lambda_2 + 2\lambda_3 + 4\lambda_4 \geq 1 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

Initial columns: λ_1 and λ_3

Choose e.g., $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and (1), that is, the variables λ_1 and λ_3

$$z^*_{\text{MP}} \leq \bar{z}_{\text{RMP}} := \begin{array}{rr} \min & \lambda_3 \\ \text{s.t.} & 2\lambda_3 \geq 1 \\ & \lambda_1 + \lambda_3 = 1 \\ & \lambda_1, \lambda_3 \geq 0 \end{array} \quad = \quad \begin{array}{rr} \max & \pi + q \\ \text{s.t.} & q \leq 0 \\ & 2\pi + q \leq 1 \\ & \pi \geq 0 \end{array}$$

Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_3) = (\frac{1}{2}, \frac{1}{2}) \Longrightarrow \widehat{\boldsymbol{x}} = (\frac{1}{2}, 0)^\top$, $\widehat{\pi} = \frac{1}{2}$, $\widehat{q} = 0$
Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(0,1)\boldsymbol{x}\} = 0 \Longrightarrow$ Optimum for (MP)

Fix variable values: $x_1 = 0$ or $x_1 = 1$

$$\begin{array}{ccc} x_1 = 0 & & x_1 = 1 \\ \Downarrow & & \Downarrow \\ \lambda_3 = 0 & & \lambda_1 = 0 \end{array}$$

Branching, left (IMP0): $\lambda_3 = 0$

$$\begin{array}{rl} \min & 0 \\ \text{s.t.} & 0 \geq 1 \\ & \lambda_1 = 1 \\ & \lambda_1 \geq 0 \end{array} \Longrightarrow \left[\begin{array}{c} \text{infeasible} \\ \Downarrow \\ \text{add} \\ \text{column} \end{array} \right] \Longrightarrow \begin{array}{rl} z_{\text{IMP0}} \leq \min & 2\lambda_2 \\ \text{s.t.} & 2\lambda_2 \geq 1 \\ & \lambda_1 + \lambda_2 = 1 \\ & \lambda_1, \lambda_2 \geq 0 \end{array}$$

$$= \quad \begin{array}{rr} \max & \pi + q \\ \text{s.t.} & q \leq 0 \\ & 2\pi + q \leq 2 \\ & \pi \geq 0 \end{array}$$

Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_2) = (\frac{1}{2}, \frac{1}{2})$
$\Longrightarrow \widehat{\mathbf{x}} = (0, \frac{1}{2})^\top$
$\widehat{\pi} = 1, \quad \widehat{q} = 0$

Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(-1, 0)\mathbf{x} - 0\} = -1 < 0$

$\Longrightarrow$ New column! (λ_3 or λ_4, but $\lambda_3 \equiv 0$) $\Longrightarrow$ Choose λ_4

$$z_{\text{IMP0}} \leq \min \quad 2\lambda_2 + 3\lambda_4 \qquad = \max \quad \pi + q$$

$$\begin{aligned} \text{s.t.} \quad 2\lambda_2 + 4\lambda_4 &\geq 1 \\ \lambda_1 + \lambda_2 + \lambda_4 &= 1 \\ \lambda_1, \lambda_2, \lambda_4 &\geq 0 \end{aligned} \qquad \begin{aligned} \text{s.t.} \quad q &\leq 0 \\ 2\pi + q &\leq 2 \\ 4\pi + q &\leq 3 \\ \pi &\geq 0 \end{aligned}$$

- Solution: $(\widehat{\lambda}_1, \widehat{\lambda}_3, \widehat{\lambda}_4) = (\frac{3}{4}, 0, \frac{1}{4}) \Longrightarrow \widehat{\boldsymbol{x}} = (\frac{1}{4}, \frac{1}{4})^\top$, $\widehat{\pi} = \frac{3}{4}$, $\widehat{q} = 0$
- Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(-\frac{1}{2}, \frac{1}{2})\boldsymbol{x}\} = -\frac{1}{2} \Longrightarrow$
- Generate new column: λ_3, but $\lambda_3 \equiv 0 \Longrightarrow$ Optimum for (IMP0)

Branching, right (IMP1): $\lambda_1 = 0$

$$z_{\text{IMP1}} \leq \min \ \lambda_3 \quad \text{s.t.} \ 2\lambda_3 \geq 1, \ \lambda_3 = 1, \ \lambda_3 \geq 0 \quad = \quad \max \ \pi + q \quad \text{s.t.} \ 2\pi + q \leq 1, \ \pi \geq 0$$

- Solution: $\widehat{\lambda}_3 = 1 \implies \widehat{\boldsymbol{x}} = (1,0)^\top$, $\widehat{\pi} = 0$, $\widehat{q} = 1$
- Reduced costs: $\min_{\mathbf{x} \in [0,1]^2} \{(1,2)\boldsymbol{x} - 1\} = -1 < 0 \implies$
- Generate new column: λ_1, but $\lambda_1 \equiv 0 \implies$ Optimum for (IMP1)

Branching, left, left: (IMP00) $\lambda_2 = \lambda_4 = 0$

(IMP00): $\lambda_2 = \lambda_3 = \lambda_4 = 0 \implies$ infeasible

Branching, left, right: (IMP01) $\lambda_1 = 0$

(IMP01): $\lambda_1 = \lambda_3 = 0$

$$z_{\text{IMP01}} \leq \min \; 2\lambda_2 + 3\lambda_4 \quad \text{s.t.} \; 2\lambda_2 + 4\lambda_4 \geq 1, \; \lambda_2 + \lambda_4 = 1, \; \lambda_2, \lambda_4 \geq 0$$

$$= \max \; \pi + q \quad \text{s.t.} \; 2\pi + q \leq 2, \; 4\pi + q \leq 3, \; \pi \geq 0$$

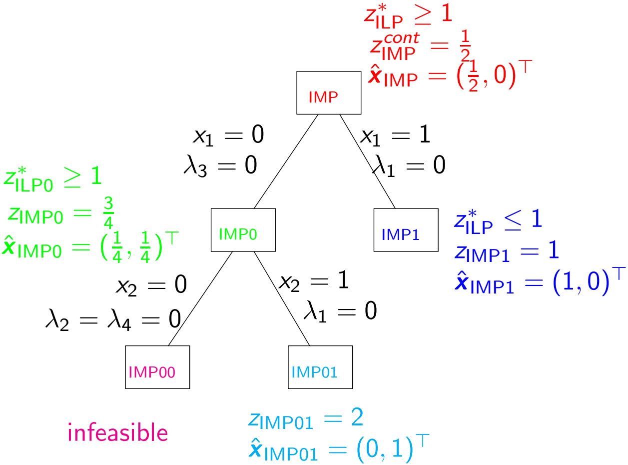
- Solution: $(\widehat{\lambda}_2, \widehat{\lambda}_4) = (1, 0)^\top \implies \widehat{\boldsymbol{x}} = (0, 1)^\top$, $\widehat{\pi} = 0$, $\widehat{q} = 2$
- Reduced costs: $\min_{\boldsymbol{x} \in [0,1]^2} \{(1, 2)\boldsymbol{x} - 2\} = -2 < 0$

  $\implies$ Generate new column: λ_1, but $\lambda_1 \equiv 0$

  $\implies$ Generate new column: λ_3, but $\lambda_3 \equiv 0$

  $\implies$ Optimum for (IMP01)

Branch–and–price tree



Illustration