

TMA521/MMA511  
Large Scale Optimization  
Lecture 8  
Dantzig–Wolfe decomposition, column  
generation, and branch–and–price

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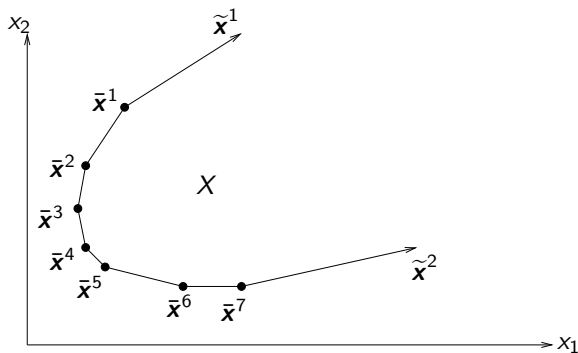
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# Column generation: a method for solving linear programs (LP) (and actually also MILP's) with very many columns (variables)

- ▶ The cutting stock problem—second formulation—comprises very many columns and relatively few rows
- ▶ What if not? (i.e., fairly many of both rows and columns, but “hard” due to integer requirements)
- ▶ The Danzig-Wolfe reformulation

# Formulation of a general linear program in a form suitable for column generation: Dantzig–Wolfe decomposition

- ▶ Let  $X = \{\mathbf{x} \in \mathbb{R}_+^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$  (or  $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ ) be a polyhedron with
- ▶ *extreme points*  $\bar{\mathbf{x}}^p$ ,  $p \in \mathcal{P}$  and
- ▶ *extreme recession directions* (extreme rays)  $\tilde{\mathbf{x}}^r$ ,  $r \in \mathcal{R}$



# Inner representation of the set $X$

$$\mathbf{x} \in X \iff \left( \begin{array}{l} \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p + \sum_{r \in \mathcal{R}} \mu_r \tilde{\mathbf{x}}^r \\ \sum_{p \in \mathcal{P}} \lambda_p = 1 \\ \lambda_p \geq 0, \quad p \in \mathcal{P} \\ \mu_r \geq 0, \quad r \in \mathcal{R} \end{array} \right)$$

- ▶  $\mathbf{x} \in X$  is a *convex* combination of the extreme points plus a *conical* combination of the extreme directions
- ▶ Use this *inner representation* of the set  $X$  to reformulate a linear program according to the *Dantzig-Wolfe decomposition principle*
- ▶ Solve by *column generation*

# A linear program and a corresponding master problem

$$\begin{aligned} \text{(LP)} \quad z^* = \text{minimum } & \mathbf{c}^\top \mathbf{x} \\ \text{subject to } & \mathbf{D}\mathbf{x} = \mathbf{d} \quad \longleftarrow \quad (\text{complicating constraints}) \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \quad \longleftarrow \quad (\text{"simple" constraints}) \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

- ▶ Let  $X = \{ \mathbf{x} \geq 0 \mid \mathbf{A}\mathbf{x} = \mathbf{b} \}$
- ▶ Extreme points of  $X$ :  $\bar{\mathbf{x}}^p, p \in \mathcal{P}$
- ▶ Extreme directions of  $X$ :  $\tilde{\mathbf{x}}^r, r \in \mathcal{R}$



# The complete master problem (MP)

$$\begin{aligned} \text{(MP)} \quad z^* = \min_{(\lambda, \mu)} \quad & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{c}^\top \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{c}^\top \tilde{\mathbf{x}}^r) \\ \text{s.t.} \quad & \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{D} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} & | \quad \pi \\ & \sum_{p \in \mathcal{P}} \lambda_p = 1 & | \quad q \\ & \lambda_p \geq 0, \quad p \in \mathcal{P} \\ & \mu_r \geq 0, \quad r \in \mathcal{R} \end{aligned}$$

- ▶ # constraints in (MP) equals # constraints in " $\mathbf{D}\mathbf{x} = \mathbf{d}$ " + 1
- ▶ # columns very large  
(equals # extreme points & # directions of  $X$ )

# The restricted master problem (RMP)

- ▶ Assume that **not all** extreme points/directions have been found:  $\bar{\mathcal{P}} \subseteq \mathcal{P}$ ;  $\bar{\mathcal{R}} \subseteq \mathcal{R}$

$$\begin{aligned} \text{(RMP)} \quad z^* \leq \min_{(\lambda, \mu)} \quad & \sum_{p \in \bar{\mathcal{P}}} \lambda_p (\mathbf{c}^\top \bar{\mathbf{x}}^p) + \sum_{r \in \bar{\mathcal{R}}} \mu_r (\mathbf{c}^\top \tilde{\mathbf{x}}^r) \\ \text{s.t.} \quad & \sum_{p \in \bar{\mathcal{P}}} \lambda_p (\mathbf{D} \bar{\mathbf{x}}^p) + \sum_{r \in \bar{\mathcal{R}}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} && | \pi \\ & \sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 && | q \\ & \lambda_p \geq 0, \quad p \in \bar{\mathcal{P}} \\ & \mu_r \geq 0, \quad r \in \bar{\mathcal{R}} \end{aligned}$$

- ▶ The number of constraints in (RMP) equals that of (MP)
- ▶ The number of columns is considerably smaller

# The linear programming dual of the (restricted) master problem (D-RMP)

- ▶ The linear programming dual of (RMP) is given by

$$\begin{aligned} \text{(D-RMP)} \quad z^* \leq \max_{(\pi, q)} \quad & \mathbf{d}^\top \pi + q \\ \text{s.t.} \quad & (\mathbf{D}\bar{\mathbf{x}}^p)^\top \pi + q \leq (\mathbf{c}^\top \bar{\mathbf{x}}^p), \quad p \in \bar{\mathcal{P}} \quad | \lambda_p \\ & (\mathbf{D}\tilde{\mathbf{x}}^r)^\top \pi \leq (\mathbf{c}^\top \tilde{\mathbf{x}}^r), \quad r \in \bar{\mathcal{R}} \quad | \mu_r \end{aligned}$$

with solution  $(\bar{\pi}, \bar{q})$

- ▶ Reduced cost for the variable  $\lambda_p$ ,  $p \in \mathcal{P} \setminus \bar{\mathcal{P}} \quad (\Leftrightarrow \bar{\mathbf{x}}^p)$ :  
$$(\mathbf{c}^\top \bar{\mathbf{x}}^p) - (\mathbf{D}\bar{\mathbf{x}}^p)^\top \bar{\pi} - \bar{q} = (\mathbf{c} - \mathbf{D}^\top \bar{\pi})^\top \bar{\mathbf{x}}^p - \bar{q}$$
- ▶ Reduced cost for the variable  $\mu_r$ ,  $r \in \mathcal{R} \setminus \bar{\mathcal{R}} \quad (\Leftrightarrow \tilde{\mathbf{x}}^r)$ :  
$$(\mathbf{c}^\top \tilde{\mathbf{x}}^r) - (\mathbf{D}\tilde{\mathbf{x}}^r)^\top \bar{\pi} = (\mathbf{c} - \mathbf{D}^\top \bar{\pi})^\top \tilde{\mathbf{x}}^r$$



# Column generation

- ▶ The smallest reduced cost is found by solving the column generation subproblem to

$$\min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^\top \bar{\boldsymbol{\pi}})^\top \mathbf{x} \quad \left( \text{alt: } \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^\top \bar{\boldsymbol{\pi}})^\top \mathbf{x} - \bar{q} \right)$$
$$\iff \min \left\{ \min_{p \in \mathcal{P}} \left\{ (\mathbf{c} - \mathbf{D}^\top \bar{\boldsymbol{\pi}})^\top \bar{\mathbf{x}}^p \right\} - \bar{q} ; \min_{r \in \mathcal{R}} \left\{ (\mathbf{c} - \mathbf{D}^\top \bar{\boldsymbol{\pi}})^\top \tilde{\mathbf{x}}^r \right\} \right\}$$

- ▶ Gives as solution an extreme point,  $\bar{\mathbf{x}}^p$ , or an extreme direction  $\tilde{\mathbf{x}}^r$
- ▶ Unbounded solutions can be detected within the simplex method! How?

$\implies$  a new column in (RMP) (if the reduced cost  $< 0$ ):

- ▶ Either  $\begin{pmatrix} \mathbf{c}^\top \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$  or  $\begin{pmatrix} \mathbf{c}^\top \tilde{\mathbf{x}}^r \\ \mathbf{D} \tilde{\mathbf{x}}^r \\ 0 \end{pmatrix}$  enters the problem and improves the solution

# A small integer linear optimization example of Dantzig-Wolfe decomposition and column generation

(ILP)

$$\begin{aligned} z_{\text{ILP}}^* &= \min 2x_1 + 3x_2 + x_3 + 4x_4 \\ &\text{s.t. } 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 \\ &\quad x_1 + x_2 + x_3 + x_4 = 2 \\ &\quad x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned} \quad | \quad \mathbf{D}\mathbf{x} = \mathbf{d}$$

$$\begin{aligned} \blacktriangleright X_{\text{ILP}} &= \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \\ &= \{\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^6\} \end{aligned}$$

$\blacktriangleright$  Optimal solution:  $\mathbf{x}_{\text{ILP}}^* = (0, 1, 1, 0)^\top$

$\blacktriangleright$  Optimal value:  $z_{\text{ILP}}^* = 4$

# Linear programming relaxation

(LP)

$$\begin{aligned} z^* = \min & \quad 2x_1 + 3x_2 + x_3 + 4x_4 & [c^T x] \\ \text{s.t.} & \quad 3x_1 + 2x_2 + 3x_3 + 2x_4 = 5 & [Dx = d] \\ & \quad x_1 + x_2 + x_3 + x_4 = 2 & [x \in X] \\ & \quad 0 \leq x_1, x_2, x_3, x_4 \leq 1 & [x \in X] \end{aligned}$$

$$\begin{aligned} \blacktriangleright X &= \text{conv} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \\ &= \text{conv} \{ \bar{x}^1, \dots, \bar{x}^6 \} \\ &= \left\{ x \in \mathbb{R}^4 \mid x = \sum_{p=1}^6 \lambda_p \bar{x}^p; \sum_{p=1}^6 \lambda_p = 1; \lambda_p \geq 0, p = 1, \dots, 6 \right\} \end{aligned}$$

# The complete master problem and the initial columns

(MP)

$$\begin{aligned} z^* = \min \quad & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 + 7\lambda_5 + 5\lambda_6 \\ \text{s.t.} \quad & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 + 4\lambda_5 + 5\lambda_6 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 0 \end{aligned}$$

- ▶ Initial columns:  $\lambda_1, \lambda_2, \lambda_3$

(RMP)

$$\begin{aligned} z^* \leq \min \quad & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 \\ \text{s.t.} \quad & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}$$

(D-RMP)

$$\begin{aligned} z^* \leq \max \quad & 5\pi + q \\ \text{s.t.} \quad & 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \end{aligned}$$

- ▶ Solution:  $\bar{\lambda} = (1, 0, 0)^\top, \quad \bar{\pi} = -2, \quad \bar{q} = 15$

$$\begin{aligned}\min_{\mathbf{x} \in X} \left\{ (\mathbf{c} - \mathbf{D}^\top \bar{\pi})^\top \mathbf{x} - \bar{q} \right\} &= \min_{p=1, \dots, 6} \left\{ (\mathbf{c} - \mathbf{D}^\top \bar{\pi})^\top \bar{\mathbf{x}}^p - \bar{q} \right\} \\ &= \min_{p=1, \dots, 6} \left\{ [(2, 3, 1, 4) - (3, 2, 3, 2) \cdot (-2)] \bar{\mathbf{x}}^p - 15 \right\} \\ &= \min \{0, 0, 1, -1, 0, 0\} = -1 < 0\end{aligned}$$

- ▶ New extreme point found in (LP):  $\bar{\mathbf{x}}^4 = (0, 1, 1, 0)^\top$

$$\Rightarrow \text{New column in (RMP): } \begin{pmatrix} \mathbf{c}^\top \bar{\mathbf{x}}^4 \\ \mathbf{D} \bar{\mathbf{x}}^4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$$

# New, extended restricted master problem

(RMP)

$$\begin{aligned} z^* \leq \min & 5\lambda_1 + 3\lambda_2 + 6\lambda_3 + 4\lambda_4 \\ \text{s.t.} & 5\lambda_1 + 6\lambda_2 + 5\lambda_3 + 5\lambda_4 = 5 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

(D-RMP)

$$\begin{aligned} z^* \leq \max & 5\pi + q \\ \text{s.t.} & 5\pi + q \leq 5 \\ & 6\pi + q \leq 3 \\ & 5\pi + q \leq 6 \\ & 5\pi + q \leq 4 \end{aligned}$$

► Solution:  $\bar{\lambda} = (0, 0, 0, 1)^\top$ ,  $\bar{\pi} = -1$ ,  $\bar{q} = 9$

► Reduced costs:

$$\min_{p=1, \dots, 6} \{ (5, 5, 4, 6) \bar{x}^p - 9 \} = \min \{ 1, 0, 2, 0, 2, 1 \} = 0$$

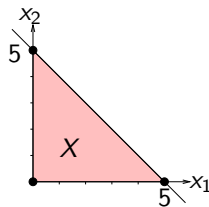
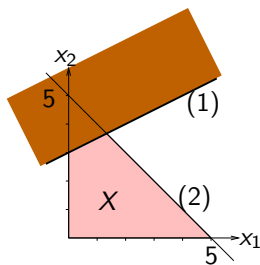
# Optimal solution to (MP) and to (LP)

- ▶  $\lambda^* = (0, 0, 0, 1, 0, 0)^\top$ ,  $\pi^* = -1$ ,  $q^* = 9$
- ⇒  $\mathbf{x}^* = \bar{\mathbf{x}}^4 = (0, 1, 1, 0)^\top = \mathbf{x}_{\text{ILP}}^*$ ,  $z^* = 4 = z_{\text{ILP}}^*$
- ▶ A coincidence that the solution was integral!
- ▶ In general, the solution  $\mathbf{x}^*$  to (LP) may have fractional variable values
- ▶ Solution to (ILP)?
- ▶ Need to find an integer solution (not certainly an optimal solution to (ILP), why?) among the columns generated, i.e., solve

$$\min \left\{ (2, 3, 1, 4)\mathbf{x} \mid (3, 2, 3, 2)\mathbf{x} = 5, \mathbf{x} \in \{\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \bar{\mathbf{x}}^3, \bar{\mathbf{x}}^4\} \right\}$$

# Another numerical example of Dantzig-Wolfe decomposition and column generation

$$\begin{array}{llll} \min & x_1 - 3x_2 & & (0) \\ \text{st} & -x_1 + 2x_2 \leq 6 & (1) & \longleftarrow \text{(complicating)} \\ & x_1 + x_2 \leq 5 & (2) & \\ & x_1, x_2 \geq 0 & (3) & \end{array}$$



$$X = \{ \mathbf{x} \in \mathbb{R}_+^2 \mid x_1 + x_2 \leq 5 \} = \text{conv} \left\{ (0,0)^\top, (0,5)^\top, (5,0)^\top \right\}$$



# Complete Dantzig-Wolfe master problem

$$\mathbf{x} \in X \iff \left\{ \begin{array}{l} \mathbf{x} = \lambda_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\lambda_3 \\ 5\lambda_2 \end{pmatrix} \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \right\}$$

$$\begin{array}{ll} \min & -15\lambda_2 + 5\lambda_3 & (0) \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 & (1) \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array}$$

- ▶ The first (RMP) is then constructed from the points  $(0, 0)^\top$  and  $(0, 5)^\top$  (corresponds to  $\lambda_1$  and  $\lambda_2$ )

# Iteration 1



$$\begin{array}{ll} \min & -15\lambda_2 & (0) \\ \text{s.t.} & 10\lambda_2 \leq 6 & (1) \\ & \lambda_1 + \lambda_2 = 1 \\ & \lambda_1, \lambda_2 \geq 0 \end{array} \quad \left| \begin{array}{l} \text{Solution:} \\ \text{Dual solution:} \end{array} \right. \quad \begin{array}{l} \bar{\lambda} = \left(\frac{2}{5}, \frac{3}{5}\right)^T \\ \bar{\pi} = -\frac{3}{2}, \bar{q} = 0 \end{array}$$

- ▶ Smallest reduced cost:

$$\begin{aligned} \min_{\mathbf{x} \in X} [(\mathbf{c}^T - \bar{\pi} \mathbf{D})\mathbf{x} - \bar{q}] &= \min_{\mathbf{x} \in X} ([ (1, -3) - (-\frac{3}{2})(-1, 2) ] \mathbf{x} - 0) \\ &= \min \{ -\frac{1}{2}x_1 \mid x_1 + x_2 \leq 5; \mathbf{x} \geq \mathbf{0}^2 \} = -\frac{5}{2} < 0 \implies \bar{\mathbf{x}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \end{aligned}$$

- ▶ New column:

$$\left. \begin{array}{l} \mathbf{c}^T \bar{\mathbf{x}} = (1, -3)(5, 0)^T = 5 \\ \mathbf{D}\bar{\mathbf{x}} = (-1, 2)(5, 0)^T = -5 \end{array} \right\} \implies \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$$

## Iteration 2

$$\begin{array}{l|l} \min & -15\lambda_2 + 5\lambda_3 \\ \text{s.t.} & 10\lambda_2 - 5\lambda_3 \leq 6 \\ & \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{array} \quad \left| \quad \begin{array}{l} \text{Solution:} \quad \bar{\lambda} = \left(0, \frac{11}{15}, \frac{4}{15}\right)^\top \\ \text{Dual solution:} \quad \bar{\pi} = -\frac{4}{3}, \bar{q} = -\frac{5}{3} \end{array} \right.$$

- ▶ Smallest reduced cost:

$$\begin{aligned} \min_{\mathbf{x} \in X} \left[ (\mathbf{c}^\top - \bar{\pi} \mathbf{D}) \mathbf{x} - \bar{q} \right] &= \min_{\mathbf{x} \in X} \left( [(1, -3) - (-\frac{4}{3})(-1, 2)] \mathbf{x} - (-\frac{5}{3}) \right) \\ &= \min \left\{ -\frac{1}{3}x_1 - \frac{1}{3}x_2 + \frac{5}{3} \mid x_1 + x_2 \leq 5; \mathbf{x} \geq \mathbf{0}^2 \right\} = 0 \end{aligned}$$

- ▶ Optimal solution:  $\lambda^* = \left(0, \frac{11}{15}, \frac{4}{15}\right)^\top$

$$\implies \mathbf{x}^* = (5\lambda_3, 5\lambda_2)^\top = \left(\frac{4}{3}, \frac{11}{3}\right)^\top; \quad z^* = \frac{4}{3} - 3 \cdot \frac{11}{3} = -9\frac{2}{3}$$

# Upper bound on the optimal objective value for (LP)

- ▶ The complete master problem (MP):

$$z^* = \min \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{c}^\top \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{c}^\top \tilde{\mathbf{x}}^r)$$

$$\text{s.t. } \sum_{p \in \mathcal{P}} \lambda_p (\mathbf{D} \bar{\mathbf{x}}^p) + \sum_{r \in \mathcal{R}} \mu_r (\mathbf{D} \tilde{\mathbf{x}}^r) = \mathbf{d} \quad | \quad \boldsymbol{\pi}$$
$$\sum_{p \in \mathcal{P}} \lambda_p = 1 \quad | \quad q$$

$$\lambda_p, \mu_r \geq 0, \quad p \in \mathcal{P}, r \in \mathcal{R}$$

- ▶ The (dual of the) restricted master problem (D-RMP) yields an upper bound on  $z^*$ :

$$z^* \leq \bar{z} = \mathbf{d}^\top \bar{\boldsymbol{\pi}} + \bar{q} =: \max_{(\boldsymbol{\pi}, q)} \mathbf{d}^\top \boldsymbol{\pi} + q$$

$$\text{s.t. } (\mathbf{D} \bar{\mathbf{x}}^p)^\top \boldsymbol{\pi} + q \leq \mathbf{c}^\top \bar{\mathbf{x}}^p, \quad p \in \bar{\mathcal{P}}$$

$$(\mathbf{D} \tilde{\mathbf{x}}^r)^\top \boldsymbol{\pi} \leq \mathbf{c}^\top \tilde{\mathbf{x}}^r, \quad r \in \bar{\mathcal{R}}$$

# Lower bound on the optimal objective value for (LP)

- ▶ Let  $\lambda_p^*$ ,  $p \in \mathcal{P}$ , and  $\mu_r^*$ ,  $r \in \mathcal{R}$ , be optimal in the complete master problem (MP)
- ▶ Let  $(\bar{\pi}, \bar{q})$  be an optimal dual solution for the restricted master problem (RMP), with columns corresponding to  $\bar{\mathcal{P}}$  and  $\bar{\mathcal{R}}$
- ▶ Multiply the right-hand side elements of the primal (i.e.,  $\mathbf{d}$  and 1) by  $\bar{\pi}$  and  $\bar{q}$ , respectively

⇒ (1'st ineq. from dual, 2'nd ineq. since  $\sum \lambda_p^* = 1$ )

$$\begin{aligned} 0 &\geq z^* - \bar{z} = z^* - \mathbf{d}^\top \bar{\pi} - 1 \cdot \bar{q} \\ &= \sum_{p \in \mathcal{P}} \lambda_p^* \left[ \mathbf{c}^\top \bar{\mathbf{x}}^p - (\mathbf{D}\bar{\mathbf{x}}^p)^\top \bar{\pi} - \bar{q} \right] + \sum_{r \in \mathcal{R}} \mu_r^* \left[ \mathbf{c}^\top \tilde{\mathbf{x}}^r - (\mathbf{D}\tilde{\mathbf{x}}^r)^\top \bar{\pi} \right] \\ &\geq \min_{p \in \mathcal{P}} \left[ \mathbf{c}^\top \bar{\mathbf{x}}^p - (\mathbf{D}\bar{\mathbf{x}}^p)^\top \bar{\pi} - \bar{q} \right] + \sum_{r \in \mathcal{R}} \mu_r^* \min_{s \in \mathcal{R}} \left[ \mathbf{c}^\top \tilde{\mathbf{x}}^s - (\mathbf{D}\tilde{\mathbf{x}}^s)^\top \bar{\pi} \right] \end{aligned}$$

# Lower bound on the optimal objective value for (LP)

- ▶ If the subproblem has an unbounded solution then no optimistic estimate (i.e., lower bound since we minimize) can be computed in this iteration
- ▶ Otherwise it holds that

$$\min_{s \in \mathcal{R}} \left[ \mathbf{c}^\top \tilde{\mathbf{x}}^s - (\mathbf{D}\tilde{\mathbf{x}}^s)^\top \bar{\boldsymbol{\pi}} \right] \geq 0$$

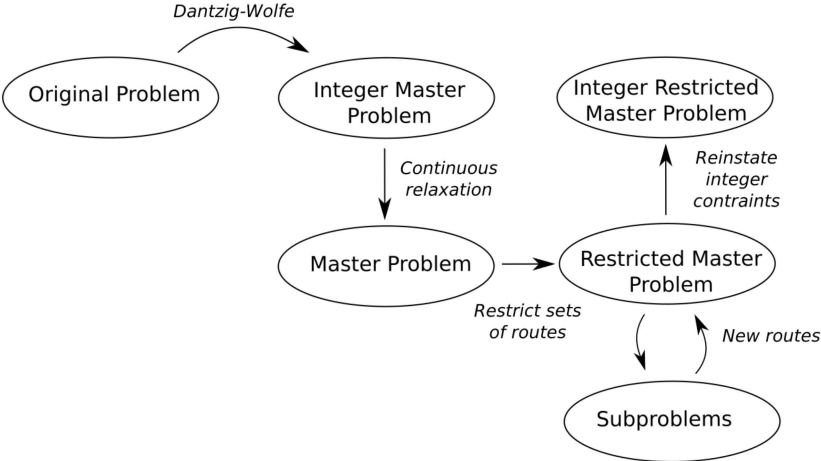
⇒

$$\begin{aligned} \bar{z} \geq z^* &\geq \bar{z} + \min_{p \in \mathcal{P}} \left[ (\mathbf{c} - \mathbf{D}^\top \bar{\boldsymbol{\pi}})^\top \bar{\mathbf{x}}^p - \bar{q} \right] \\ &= \bar{z} + \min_{\mathbf{x} \in X} (\mathbf{c} - \mathbf{D}^\top \bar{\boldsymbol{\pi}})^\top \mathbf{x} - \bar{q} \\ &=: \underline{z} \end{aligned}$$

# Convergence of the column generation algorithm

- ▶ The number of columns generated is finite, since  $X$  is polyhedral
  - ▶ When no more columns are generated, the solution to the last restricted master problem will also solve (LP)
  - ▶ For each new column that is added to the restricted master problem (RMP), its optimal objective value will decrease
- ⇒ The pessimistic estimate  $\bar{z}_k$  converges monotonically to  $z^*$
- ▶ The optimistic estimate  $\underline{z}_k$  also converges, but not monotonically
  - ▶ If at iteration  $k$  an optimal solution to the complete master problem (MP) is received, then  $\underline{z}_k = \bar{z}_k$  holds
- ⇒ Stopping criterion:  $\bar{z}_k - \underline{z}_k^* \leq \varepsilon$ , where  $\underline{z}_k^* = \max_{s=1, \dots, k} \underline{z}_s$  and  $\varepsilon > 0$

# Column generation





# Dantzig-Wolfe decomposition applied to a linear program with block-angular structure

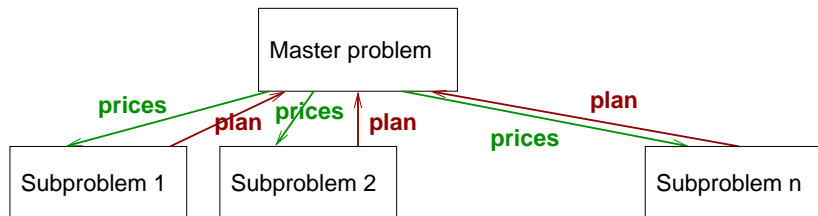
$$\begin{array}{ll} \text{(LP)} \quad \max & \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 + \cdots + \mathbf{c}_n^\top \mathbf{x}_n \\ \text{s.t.} & \mathbf{D}_1 \mathbf{x}_1 + \mathbf{D}_2 \mathbf{x}_2 + \cdots + \mathbf{D}_n \mathbf{x}_n \leq \mathbf{d} \quad | \quad \text{Dual var: } \boldsymbol{\pi} \\ & \mathbf{A}_1 \mathbf{x}_1 \leq \mathbf{b}_1 \quad | \quad \mathbf{x}_1 \in X_1 \\ & \mathbf{A}_2 \mathbf{x}_2 \leq \mathbf{b}_2 \quad | \quad \mathbf{x}_2 \in X_2 \\ & \quad \quad \quad \dots \quad \quad \quad \cdot \cdot \\ & \quad \quad \quad \mathbf{A}_n \mathbf{x}_n \leq \mathbf{b}_n \quad | \quad \mathbf{x}_n \in X_n \\ & \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \geq \mathbf{0} \end{array}$$

Cartesian product set:

$$X = X_1 \times X_2 \times \dots \times X_n$$

# Dantzig-Wolfe decomposition interpreted as decentralized planning

- ▶ The main office (master problem) sets prizes ( $\pi$ ) for the common resources (complicating constraints)
- ▶ The departments (subproblems) suggest (production) plans (i.e., columns) ( $D_j \bar{x}_j^p$ ) based on given prices
- ▶ The main office “mixes” the suggested plans (columns) optimally; sets new prices
- ▶ The procedure is repeated



# Inner representations of the sets $X_j, j = 1, \dots, n$

- ▶ Let  $X_j = \{\mathbf{x}_j \geq \mathbf{0} \mid \mathbf{A}_j \mathbf{x}_j \leq \mathbf{b}_j\}$  and express  $\mathbf{x}_j$  as

$$\mathbf{x}_j \in X_j \iff \left( \begin{array}{l} \mathbf{x}_j = \sum_{p \in \mathcal{P}_j} \lambda_{pj} \bar{\mathbf{x}}_j^p + \sum_{r \in \mathcal{R}_j} \mu_{rj} \tilde{\mathbf{x}}_j^r \\ \sum_{p \in \mathcal{P}_j} \lambda_{pj} = 1 \\ \lambda_{pj} \geq 0, \quad p \in \mathcal{P}_j \\ \mu_{rj} \geq 0, \quad r \in \mathcal{R}_j \end{array} \right) \quad j = 1, \dots, n$$

$$\iff$$

- ▶  $X_j = \text{conv}\{\bar{\mathbf{x}}_j^p \mid p \in \mathcal{P}_j\} + \text{cone}\{\tilde{\mathbf{x}}_j^r \mid r \in \mathcal{R}_j\}, \quad j = 1, \dots, n$

# The complete master problem (MP) (here: maximization)

$$(MP) \quad \max_{(\lambda, \mu)} \sum_{j=1}^n \left( \sum_{p \in \mathcal{P}_j} \lambda_{pj} (\mathbf{c}^\top \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}_j} \mu_{rj} (\mathbf{c}^\top \tilde{\mathbf{x}}_j^r) \right)$$

$$\text{s.t.} \quad \sum_{j=1}^n \left( \sum_{p \in \mathcal{P}_j} \lambda_{pj} (\mathbf{D} \bar{\mathbf{x}}_j^p) + \sum_{r \in \mathcal{R}_j} \mu_{rj} (\mathbf{D} \tilde{\mathbf{x}}_j^r) \right) \leq \mathbf{d}$$

$$\sum_{p \in \mathcal{P}_j} \lambda_{pj} = 1, \quad j = 1, \dots, n$$

$$\lambda_{pj} \geq 0, \quad p \in \mathcal{P}_j, \quad j = 1, \dots, n$$

$$\mu_{rj} \geq 0, \quad r \in \mathcal{R}_j, \quad j = 1, \dots, n$$

- ▶ The number of constraints in the master problem equals the number of constraints in " $\mathbf{D}\mathbf{x} = \mathbf{d}$ " +  $n$
- ▶ The restricted master problem (RMP) is formulated analogously as before (with  $\bar{\mathcal{P}}_j \subseteq \mathcal{P}_j$  and  $\bar{\mathcal{R}}_j \subseteq \mathcal{R}_j$ ,  $j=1, \dots, n$ )

# Use the linear programming dual solution to generate new columns

- ▶ The linear programming dual of the restricted master problem

$$\begin{aligned} \text{(D-RMP)} \quad \min_{(\boldsymbol{\pi}, \mathbf{q})} \quad & \mathbf{d}^\top \boldsymbol{\pi} + \sum_{j=1}^n q_j \\ \text{s.t.} \quad & (\mathbf{D}_j \bar{\mathbf{x}}_j^p)^\top \boldsymbol{\pi} + q_j \geq (\mathbf{c}_j^\top \bar{\mathbf{x}}_j^p), \quad p \in \bar{\mathcal{P}}_j, \quad j = 1, \dots, n \quad | \quad \lambda_{pj} \\ & (\mathbf{D}_j \tilde{\mathbf{x}}_j^r)^\top \boldsymbol{\pi} \geq (\mathbf{c}_j^\top \tilde{\mathbf{x}}_j^r), \quad r \in \tilde{\mathcal{R}}_j, \quad j = 1, \dots, n \quad | \quad \mu_{rj} \end{aligned}$$

with solution  $\bar{\boldsymbol{\pi}}, \bar{q}_j, j = 1, \dots, n$

- ▶ Generate new columns (maximization  $\Leftrightarrow$  reduced cost  $> 0$ ):

For  $j = 1, \dots, n$ , solve  $\max_{\mathbf{x}_j \in X_j} \left\{ (\mathbf{c}_j - \mathbf{D}_j^\top \bar{\boldsymbol{\pi}})^\top \mathbf{x}_j - \bar{q}_j \right\}$

$\Rightarrow$  Subproblem solutions:  $\bar{\mathbf{x}}_j^p$  or  $\tilde{\mathbf{x}}_j^r$

$\Rightarrow$  New columns:  $\left( \begin{array}{c} \mathbf{c}_j^\top \bar{\mathbf{x}}_j^p \\ \mathbf{D}_j^\top \bar{\mathbf{x}}_j^p \\ 1 \end{array} \right)$  or  $\left( \begin{array}{c} \mathbf{c}_j^\top \tilde{\mathbf{x}}_j^r \\ \mathbf{D}_j^\top \tilde{\mathbf{x}}_j^r \\ 0 \end{array} \right)$  for  $j = 1, \dots, n$

# Find feasible solutions (right-hand side allocation)

- ▶ If the (RMP) is not solved to optimality, let  $\bar{\lambda}_{pj}$ ,  $p \in \bar{\mathcal{P}}_j$ , and  $\bar{\mu}_{rj}$ ,  $r \in \bar{\mathcal{R}}_j$ ,  $j = 1, \dots, n$ , denote a **feasible** (and almost optimal) solution to (RMP). It then holds that

$$\sum_{j=1}^n \mathbf{D}_j \underbrace{\left( \sum_{p \in \bar{\mathcal{P}}_j} \bar{\lambda}_{pj} \bar{\mathbf{x}}_j^p + \sum_{r \in \bar{\mathcal{R}}_j} \bar{\mu}_{rj} \tilde{\mathbf{x}}_j^r \right)}_{\in X_j} \leq \mathbf{d} \quad (*)$$

- ▶ If the (RMP) solution is close to optimal, a good feasible solution to (LP) is given by a solution, for  $j = 1, \dots, n$ , to

$$\begin{aligned} & \max \mathbf{c}_j^\top \mathbf{x}_j \\ & \text{s.t. } \mathbf{D}_j \mathbf{x}_j \leq \sum_{p \in \bar{\mathcal{P}}_j} \bar{\lambda}_{pj} (\mathbf{D}_j \bar{\mathbf{x}}_j^p) + \sum_{r \in \bar{\mathcal{R}}_j} \bar{\mu}_{rj} (\mathbf{D}_j \tilde{\mathbf{x}}_j^r) \quad (**) \\ & \mathbf{x}_j \in X_j \end{aligned}$$

where (\*) and (\*\*)  $\implies \sum_{j=1}^n \mathbf{D}_j \mathbf{x}_j \leq \mathbf{d}$  holds

# Branch-and-price for integer linear optimization problems

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For ease of notation, we here consider binary variables

$$\begin{aligned} \text{(ILP)} \quad z_{\text{ILP}}^* &= \min \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad \mathbf{D}\mathbf{x} &= \mathbf{d} \\ \mathbf{x} \in X &= \{\mathbf{x} \in \mathbb{B}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\} = \{\bar{\mathbf{x}}^p \mid p \in \mathcal{P}\} \end{aligned}$$

- ▶ Inner representation (and convexification):

$$\text{conv } X = \left\{ \mathbf{x} = \sum_{p \in \mathcal{P}} \lambda_p \bar{\mathbf{x}}^p \mid \sum_{p \in \mathcal{P}} \lambda_p = 1; \lambda_p \geq 0, p \in \mathcal{P} \right\}$$

- ▶ Define  $\mathbf{c}_p := \mathbf{c}^\top \bar{\mathbf{x}}^p$  and  $\mathbf{d}_p := \mathbf{D}\bar{\mathbf{x}}^p$ ,  $p \in \mathcal{P}$



# A stronger formulation: The complete integer master problem (IMP)

$$\begin{aligned} \text{(IMP)} \quad z_{\text{ILP}}^* = z_{\text{IMP}}^* := & \min \sum_{p \in \mathcal{P}} c_p \lambda_p \\ \text{s.t.} \quad & \sum_{p \in \mathcal{P}} \mathbf{d}_p \lambda_p = \mathbf{d} \\ & \sum_{p \in \mathcal{P}} \lambda_p = 1 \\ & \lambda_p \in \{0, 1\}, \quad p \in \mathcal{P} \end{aligned}$$

# The complete master problem (MP)

- ▶ The continuous relaxation of (IMP) to (MP) (i.e., relax  $\lambda_p \in \{0, 1\}$  to  $\lambda_p \geq 0$ ) yields the same lower bound as the Lagrangian dual w.r.t. the constraints  $\mathbf{D}\mathbf{x} = \mathbf{d}$  (i.e.,  $z_L^*$ )

$$\begin{aligned} \text{(MP)} \quad z_{\text{IMP}}^* \geq z_{\text{MP}}^* &:= \min \sum_{p \in \mathcal{P}} c_p \lambda_p \\ \text{s.t.} \quad \sum_{p \in \mathcal{P}} \mathbf{d}_p \lambda_p &= \mathbf{d} \\ \sum_{p \in \mathcal{P}} \lambda_p &= 1 \\ \lambda_p &\geq 0, \quad p \in \mathcal{P} \end{aligned}$$

- ▶ The continuous relaxation (LP) of (ILP) is never better than any Lagrangian dual bound
- $\Rightarrow z_{\text{LP}}^* \leq z_L^* = z_{\text{MP}}^* \leq z_{\text{IMP}}^* = z_{\text{ILP}}^*$

# Restricted (continuous) master problem (RMP)

- ▶ Let  $\bar{\mathcal{P}} \subseteq \mathcal{P}$ .

(RMP) is a restriction of (MP)  $\iff$  (MP) is a relaxation of (RMP)

$$\text{(RMP)} \quad z_{\text{IMP}}^* \geq z_{\text{MP}}^* \leq \bar{z}_{\text{RMP}} := \min \sum_{p \in \bar{\mathcal{P}}} c_p \lambda_p$$

$$\text{s.t.} \quad \sum_{p \in \bar{\mathcal{P}}} \mathbf{d}_p \lambda_p = \mathbf{d}$$

$$\sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1 \quad (*)$$

$$\lambda_p \geq 0, \quad p \in \bar{\mathcal{P}}$$

- ▶ Generate columns  $\begin{pmatrix} c_p \\ \mathbf{d}_p \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{c}^\top \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$  until an (almost)

optimal solution to (MP),  $\hat{\lambda}_p$ ,  $p \in \bar{\mathcal{P}}$ , is found

- ▶ The corresponding solution to (LP):  $\hat{\mathbf{x}} = \sum_{p \in \bar{\mathcal{P}}} \hat{\lambda}_p \bar{\mathbf{x}}^p$

# Branching over a variable $x_j$ with $0 < \hat{x}_j < 1$

[left branch]

$$x_j = 0$$

or

$$x_j = 1$$

[right branch]



$$x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 0$$

$$x_j = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \bar{x}_j^p = 1$$



[delete col's]

$$\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 1} \lambda_p = 0$$

$$\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 1} \lambda_p = 1$$

[replaces (\*)]



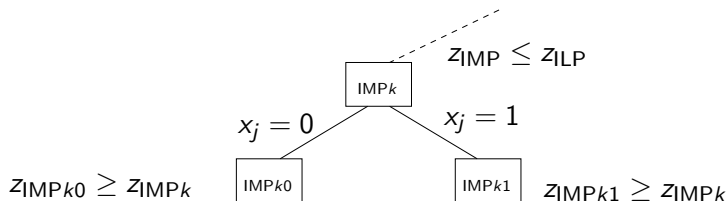
[replaces (\*)]

$$\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 0} \lambda_p = 1$$

$$\sum_{p \in \bar{\mathcal{P}}: \bar{x}_j^p = 0} \lambda_p = 0$$

[delete col's]

# Generate columns in B&B nodes



- ▶ In each node ( $IMP$ ), ( $IMP_0$ ), ( $IMP_1$ ), ...): Generate columns until it is (almost) optimal (i.e., all reduced costs  $\geq 0$ ) or it is verified infeasible
  - ▶ If  $\mathbf{x}_{IMP_{kl\dots}}^*$  feasible  $\implies z_{IMP_{kl\dots}}^* \geq z_{ILP}^* \implies$  Cut off the branch ( $k, l, \dots$ )
- $\implies$  Cut branches ( $r, s, \dots$ ) with  $z_{IMP_{rs\dots}}^* \geq z_{IMP_{kl\dots}}^*$

# The column generation subproblem, reduced costs

- ▶  $\min_{\mathbf{x} \in X^k} (\mathbf{c} - \mathbf{D}^\top \hat{\boldsymbol{\pi}}^k)^\top \mathbf{x} - \hat{q}^k =: (\mathbf{c} - \mathbf{D}^\top \hat{\boldsymbol{\pi}}^k)^\top \bar{\mathbf{x}}^p - \hat{q}^k =: \bar{c}(\bar{\mathbf{x}}^p)$
- ▶  $(\hat{\boldsymbol{\pi}}^k, \hat{q}^k)$  is a dual solution to the (RMP) and  $X^k = X \cap \{\mathbf{x} \mid x_j = k\}$ ,  $k \in \{0, 1\}$  (etc. down the tree)
- ▶ If  $\bar{c}(\bar{\mathbf{x}}^p) < 0$  then  $\begin{pmatrix} \mathbf{c}^\top \bar{\mathbf{x}}^p \\ \mathbf{D} \bar{\mathbf{x}}^p \\ 1 \end{pmatrix}$  is a new column in (IMP $k$ )
- ▶ Minimization of the subproblem? A solution (column)  $\bar{\mathbf{x}}^r$  is good enough if  $\bar{c}(\bar{\mathbf{x}}^r) < 0$
- ▶ If  $\bar{c}(\bar{\mathbf{x}}^p) \geq 0$  then no more columns are needed to solve (IMP $k$ ) to optimality
- ▶ In general, several columns can be generated simultaneously, i.e., save all columns  $r$  with  $\bar{c}(\bar{\mathbf{x}}^r) < 0$  (not only the optimal)
- ▶ The same columns may be generated in different nodes  $\implies$  create “column pool” to check w.r.t. reduced costs  $\bar{c}$

# An instance solved by Branch-and-price

$$z_{\text{ILP}}^* = \min_{\substack{x_1 + 2x_2 \\ \text{s.t. } 2x_1 + 2x_2 \geq 1 \\ x_1, x_2 \in \{0, 1\}}} = z_{\text{IMP}}^* \geq z_{\text{MP}}^* \geq z_{\text{LP}}^* = \min_{\substack{x_1 + 2x_2 \\ \text{s.t. } 2x_1 + 2x_2 \geq 1 \\ 0 \leq x_1, x_2 \leq 1}}$$

$$\text{conv}X = \text{conv} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \lambda_3 + \lambda_4 \\ \lambda_2 + \lambda_4 \end{pmatrix} \mid \sum_{p=1}^4 \lambda_p = 1; \lambda_p \geq 0 \right\}$$

$$\begin{aligned} \text{(MP)} \quad z_{\text{MP}}^* = \min \quad & 2\lambda_2 + \lambda_3 + 3\lambda_4 \\ \text{s.t.} \quad & 2\lambda_2 + 2\lambda_3 + 4\lambda_4 \geq 1 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

## Initial columns: $\lambda_1$ and $\lambda_3$

Choose e.g.,  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and  $(1)$ , that is, the variables  $\lambda_1$  and  $\lambda_3$

$$\begin{aligned} z_{\text{MP}}^* \leq \bar{z}_{\text{RMP}} &:= \min && \lambda_3 && = \max && \pi + q \\ &\text{s.t.} && 2\lambda_3 \geq 1 && && \text{s.t.} && q \leq 0 \\ &&& \lambda_1 + \lambda_3 = 1 && && && 2\pi + q \leq 1 \\ &&& \lambda_1, \lambda_3 \geq 0 && && && \pi \geq 0 \end{aligned}$$

Solution:  $(\hat{\lambda}_1, \hat{\lambda}_3) = (\frac{1}{2}, \frac{1}{2}) \implies \hat{\mathbf{x}} = (\frac{1}{2}, 0)^\top$ ,  $\hat{\pi} = \frac{1}{2}$ ,  $\hat{q} = 0$

Reduced costs:  $\min_{\mathbf{x} \in [0,1]^2} \{(0, 1)\mathbf{x}\} = 0 \implies$  Optimum for (MP)

$$\begin{array}{ccc} \text{Fix variable values:} & x_1 = 0 & \text{or} & x_1 = 1 \\ & \Downarrow & & \Downarrow \\ & \lambda_3 = 0 & & \lambda_1 = 0 \end{array}$$



# Branching, left (IMP0): $\lambda_3 = 0$

$$\begin{array}{l} \min 0 \\ \text{s.t. } 0 \geq 1 \\ \lambda_1 = 1 \\ \lambda_1 \geq 0 \end{array} \implies \left[ \begin{array}{c} \text{infeasible} \\ \Downarrow \\ \text{add} \\ \text{column} \end{array} \right] \implies \begin{array}{l} z_{\text{IMP0}} \leq \min 2\lambda_2 \\ \text{s.t. } 2\lambda_2 \geq 1 \\ \lambda_1 + \lambda_2 = 1 \\ \lambda_1, \lambda_2 \geq 0 \end{array}$$

$$\begin{array}{l} = \max \pi + q \\ \text{s.t. } q \leq 0 \\ 2\pi + q \leq 2 \\ \pi \geq 0 \end{array} \quad \begin{array}{l} \text{Solution: } (\hat{\lambda}_1, \hat{\lambda}_2) = \left(\frac{1}{2}, \frac{1}{2}\right) \\ \implies \hat{\mathbf{x}} = \left(0, \frac{1}{2}\right)^\top \\ \hat{\pi} = 1, \quad \hat{q} = 0 \end{array}$$

Reduced costs:  $\min_{\mathbf{x} \in [0,1]^2} \{(-1, 0)\mathbf{x} - 0\} = -1 < 0$

$\implies$  New column! ( $\lambda_3$  or  $\lambda_4$ , but  $\lambda_3 \equiv 0$ )  $\implies$  Choose  $\lambda_4$

$$\begin{aligned}
z_{\text{IMP0}} \leq \min \quad & 2\lambda_2 + 3\lambda_4 & = \max \quad & \pi + q \\
\text{s.t.} \quad & 2\lambda_2 + 4\lambda_4 \geq 1 & \text{s.t.} \quad & q \leq 0 \\
& \lambda_1 + \lambda_2 + \lambda_4 = 1 & & 2\pi + q \leq 2 \\
& \lambda_1, \lambda_2, \lambda_4 \geq 0 & & 4\pi + q \leq 3 \\
& & & \pi \geq 0
\end{aligned}$$

- ▶ Solution:  $(\hat{\lambda}_1, \hat{\lambda}_3, \hat{\lambda}_4) = (\frac{3}{4}, 0, \frac{1}{4}) \implies \hat{\mathbf{x}} = (\frac{1}{4}, \frac{1}{4})^\top, \hat{\pi} = \frac{3}{4}, \hat{q} = 0$
- ▶ Reduced costs:  $\min_{\mathbf{x} \in [0,1]^2} \{(-\frac{1}{2}, \frac{1}{2})\mathbf{x}\} = -\frac{1}{2} \implies$
- ▶ Generate new column:  $\lambda_3$ , but  $\lambda_3 \equiv 0 \implies$  Optimum for (IMP0)

## Branching, right (IMP1): $\lambda_1 = 0$

$$\begin{aligned} z_{\text{IMP1}} \leq \min \quad & \lambda_3 \\ \text{s.t.} \quad & 2\lambda_3 \geq 1 \\ & \lambda_3 = 1 \\ & \lambda_3 \geq 0 \end{aligned} \quad = \quad \begin{aligned} \max \quad & \pi + q \\ \text{s.t.} \quad & 2\pi + q \leq 1 \\ & \pi \geq 0 \end{aligned}$$

- ▶ Solution:  $\hat{\lambda}_3 = 1 \implies \hat{\mathbf{x}} = (1, 0)^\top, \hat{\pi} = 0, \hat{q} = 1$
- ▶ Reduced costs:  $\min_{\mathbf{x} \in [0,1]^2} \{(1, 2)\mathbf{x} - 1\} = -1 < 0 \implies$
- ▶ Generate new column:  $\lambda_1$ , but  $\lambda_1 \equiv 0 \implies$  Optimum for (IMP1)

Branching, left, left: (IMP00)  $\lambda_2 = \lambda_4 = 0$

(IMP00):  $\lambda_2 = \lambda_3 = \lambda_4 = 0 \implies$  infeasible

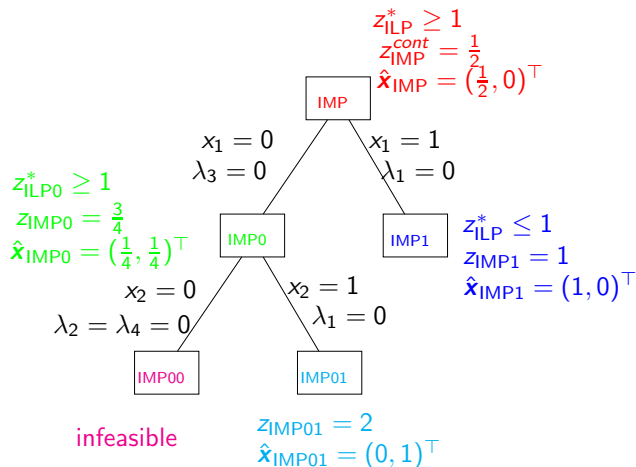
## Branching, left, right: (IMP01) $\lambda_1 = 0$

(IMP01):  $\lambda_1 = \lambda_3 = 0$

$$\begin{aligned} z_{\text{IMP01}} \leq \min \quad & 2\lambda_2 + 3\lambda_4 & = \quad & \max \quad \pi + q \\ \text{s.t.} \quad & 2\lambda_2 + 4\lambda_4 \geq 1 & & \text{s.t.} \quad 2\pi + q \leq 2 \\ & \lambda_2 + \lambda_4 = 1 & & 4\pi + q \leq 3 \\ & \lambda_2, \lambda_4 \geq 0 & & \pi \geq 0 \end{aligned}$$

- ▶ Solution:  $(\hat{\lambda}_2, \hat{\lambda}_4) = (1, 0)^\top \implies \hat{\mathbf{x}} = (0, 1)^\top, \hat{\pi} = 0, \hat{q} = 2$
- ▶ Reduced costs:  $\min_{\mathbf{x} \in [0,1]^2} \{(1, 2)\mathbf{x} - 2\} = -2 < 0$ 
  - $\implies$  Generate new column:  $\lambda_1$ , but  $\lambda_1 \equiv 0$
  - $\implies$  Generate new column:  $\lambda_3$ , but  $\lambda_3 \equiv 0$
  - $\implies$  Optimum for (IMP01)

# Branch-and-price tree



# Illustration

