TMA521/MMA511
Large Scale Optimization
Lecture 9
An instance of the cutting stock problem solved by column generation

Ann-Brith Strömberg

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Second formulation

- **Cut pattern** number j contains a_{ij} pieces of length ℓ_i
- **Feasible** pattern if $\sum_{i=1}^{m} \ell_i a_{ij} \leq L$, where $a_{ij} \geq 0$, integer
- **Variables:** $x_j =$ number of times that pattern j is used

$$\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{n} x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, && i = 1, \ldots, m \\
& x_j \geq 0, \ \text{integer}, && j = 1, \ldots, n
\end{aligned}$$

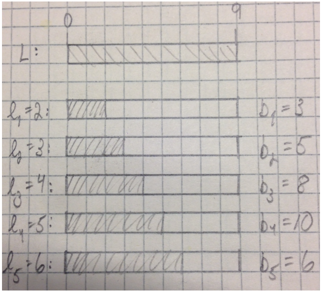
Starting solution

- Find a set of patterns that can be used to fulfill the demand for pieces of different lengths
- Natural: m unit columns (yields lots of waste) $\Longrightarrow$

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j=1}^{m} x_j \\
\text{subject to} \quad & x_j = b_j, \qquad j = 1, \ldots, m \\
& x_j \geq 0, \qquad j = 1, \ldots, m
\end{aligned}
$$

A cutting stock example

- Problem data:
  - $L = 9$
  - $(\ell_j)_{j=1}^5 = (2, 3, 4, 5, 6)$
  - $(b_j)_{j=1}^5 = (3, 5, 8, 10, 6)$
- Starting solution:
  $x_1 = 3,\ x_2 = 5,\ x_3 = 8,$
  $x_4 = 10,\ x_5 = 6$
- Totally 32 rolls are used—one for each piece



Iteration 1

Initial primal program

$$\min \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{x}$$

$$\text{s.t.} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 5 \\ 8 \\ 10 \\ 6 \end{bmatrix}$$

$$\mathbf{x} \geq \mathbf{0}$$

Solution:
$\bar{\mathbf{x}}^1 = (3, 5, 8, 10, 6)$
$\bar{z}^1 = 32$

Corresponding dual program

$$\max \quad \begin{bmatrix} 3 & 5 & 8 & 10 & 6 \end{bmatrix} \boldsymbol{\pi}$$

$$\text{s.t.} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{\pi} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution:
$\bar{\boldsymbol{\pi}}^1 = (1, 1, 1, 1, 1)$
$\bar{v}^1 = 32$

Sub-problem:

$$\max \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{a}$$

$$\text{s.t.} \quad \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \end{bmatrix} \mathbf{a} \leq 9$$

$$\mathbf{a} \geq \mathbf{0}, \text{ integer}$$

Solution:

$\bar{\mathbf{a}}^1 = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \end{bmatrix}$

Reduced cost:

$\bar{c}^1 = 1 - (\bar{\boldsymbol{\pi}}^1)^{\mathrm{T}} \bar{\mathbf{a}}^1 = -3 < 0$

Iteration 2

Primal program

$$\min \quad \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\end{bmatrix} \mathbf{x}$$

$$\text{s.t.} \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 3\\ 0 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix} \mathbf{x} = \begin{bmatrix}3\\5\\8\\10\\6\end{bmatrix}$$

$$\mathbf{x} \geq \mathbf{0}$$

Solution:

$\bar{\mathbf{x}}^2 = (0, 4, 8, 10, 6, 1)$

$\bar{z}^2 = 29$

Corresponding dual program

$$\max \quad \begin{bmatrix}3 & 5 & 8 & 10 & 6\end{bmatrix} \boldsymbol{\pi}$$

$$\text{s.t.} \begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 3 & 1 & 0 & 0 & 0\end{bmatrix} \boldsymbol{\pi} \leq \begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}$$

Solution:

$\bar{\boldsymbol{\pi}}^2 = (0, 1, 1, 1, 1)$

$\bar{v}^2 = 29$

Sub-problem:

$$\max \quad \begin{bmatrix}0 & 1 & 1 & 1 & 1\end{bmatrix} \mathbf{a}$$

$$\text{s.t.} \quad \begin{bmatrix}2 & 3 & 4 & 5 & 6\end{bmatrix} \mathbf{a} \leq 9$$

$$\mathbf{a} \geq \mathbf{0}, \ \text{integer}$$

Solution:

$\bar{\mathbf{a}}^2 = \begin{bmatrix}0 & 3 & 0 & 0 & 0\end{bmatrix}$

Reduced cost:

$\bar{c}^2 = 1 - (\bar{\boldsymbol{\pi}}^2)^{\mathrm{T}} \bar{a}^2 = -2 < 0$

Iteration 3

Primal program

$$\min \quad \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix} \mathbf{x}$$

$$\text{s.t.} \quad \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 3 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & 3\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\end{bmatrix} \mathbf{x} = \begin{bmatrix}3\\5\\8\\10\\6\end{bmatrix}$$

$$\mathbf{x} \geq \mathbf{0}$$

Solution:

$\overline{\mathbf{x}}^3 = (0, 0, 8, 10, 6, 1, \frac{4}{3})$

$\overline{z}^3 = 27$

Sub-problem:

$$\max \quad \begin{bmatrix}\frac{2}{9} & \frac{1}{3} & 1 & 1 & 1\end{bmatrix} \mathbf{a}$$

$$\text{s.t.} \quad \begin{bmatrix}2 & 3 & 4 & 5 & 6\end{bmatrix} \mathbf{a} \leq 9$$

$$\mathbf{a} \geq \mathbf{0}, \ \text{integer}$$

Corresponding dual program

$$\max \quad \begin{bmatrix}3 & 5 & 8 & 10 & 6\end{bmatrix} \boldsymbol{\pi}$$

$$\text{s.t.} \quad \begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 3 & 1 & 0 & 0 & 0\\ 0 & 3 & 0 & 0 & 0\end{bmatrix} \boldsymbol{\pi} \leq \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\end{bmatrix}$$

Solution:

$\overline{\boldsymbol{\pi}}^3 = (\frac{2}{9}, \frac{1}{3}, 1, 1, 1)$

$\overline{v}^3 = 27$

Solution:

$\overline{\mathbf{a}}^3 = \begin{bmatrix}0 & 0 & 1 & 1 & 0\end{bmatrix}$

Reduced cost:

$\overline{c}^3 = 1 - (\overline{\boldsymbol{\pi}}^3)^{\mathrm{T}} \overline{a}^3 = -1 < 0$

Iteration 4

Primal program

$$\min \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{x}$$

$$\text{s.t.} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 5 \\ 8 \\ 10 \\ 6 \end{bmatrix}$$

$$\mathbf{x} \geq \mathbf{0}$$

Solution:

$\overline{\mathbf{x}}^4 = (0, 0, 0, 2, 6, 1, \frac{4}{3}, 8)$

$\overline{z}^4 = 18\frac{1}{3}$

Sub-problem:

$$\max \quad \begin{bmatrix} \frac{2}{9} & \frac{1}{3} & 0 & 1 & 1 \end{bmatrix} \mathbf{a}$$

$$\text{s.t.} \quad \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \end{bmatrix} \mathbf{a} \leq 9$$

$$\mathbf{a} \geq \mathbf{0}, \text{ integer}$$

Corresponding dual program

$$\max \quad \begin{bmatrix} 3 & 5 & 8 & 10 & 6 \end{bmatrix} \boldsymbol{\pi}$$

$$\text{s.t.} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \boldsymbol{\pi} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution:

$\overline{\boldsymbol{\pi}}^4 = (\frac{2}{9}, \frac{1}{3}, 0, 1, 1)$, $\overline{v}^4 = 18\frac{1}{3}$

Solution:

$\overline{\mathbf{a}}^4 = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 \end{bmatrix}$

Reduced cost: $\overline{c}^4 = -\dfrac{4}{9} < 0$

Iteration 5

Primal program

$$\min \quad \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix} \mathbf{x}$$

$$\text{s.t.} \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2\\ 0 & 1 & 0 & 0 & 0 & 1 & 3 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\end{bmatrix} \mathbf{x} = \begin{bmatrix}3\\5\\8\\10\\6\end{bmatrix}$$

$$\mathbf{x} \geq \mathbf{0}$$

Solution:

$\overline{\mathbf{x}}^5 = (0, 0, 0, \frac{1}{2}, 6, 0, \frac{5}{3}, 8, \frac{3}{2})$

$\overline{z}^5 = 17\frac{2}{3}$

Corresponding dual program

$$\max \quad \begin{bmatrix}3 & 5 & 8 & 10 & 6\end{bmatrix} \boldsymbol{\pi}$$

$$\text{s.t.} \begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 3 & 1 & 0 & 0 & 0\\ 0 & 3 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\\ 2 & 0 & 0 & 1 & 0\end{bmatrix} \boldsymbol{\pi} \leq \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}$$

Solution:

$\overline{\boldsymbol{\pi}}^5 = (0, \frac{1}{3}, 0, 1, 1)$, $\overline{v}^5 = 17\frac{2}{3}$

Sub-problem:

$$\max \quad \begin{bmatrix}0 & \frac{1}{3} & 0 & 1 & 1\end{bmatrix} \mathbf{a}$$

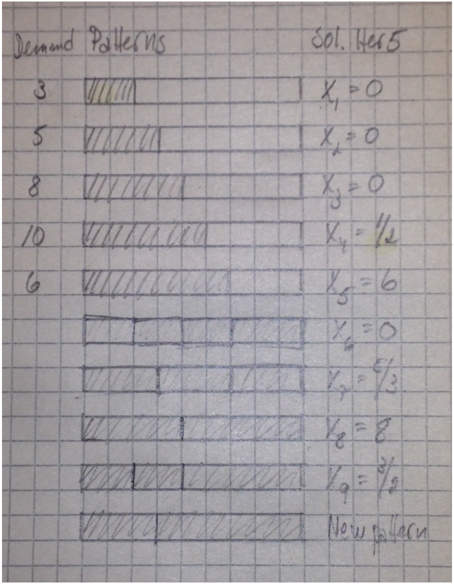
$$\text{s.t.} \quad \begin{bmatrix}2 & 3 & 4 & 5 & 6\end{bmatrix} \mathbf{a} \leq 9$$

$$\mathbf{a} \geq \mathbf{0}, \ \text{integer}$$

Solution: $\overline{\mathbf{a}}^5 = \begin{bmatrix}0 & 1 & 0 & 0 & 1\end{bmatrix}$

Reduced cost: $\overline{c}^5 = -\dfrac{1}{3} < 0$

Iteration 5


Demand
Patterns
Sol. Iter5
3
5
8
10
6
x1 = 0
x2 = 0
x3 = 0
x4 = 1/2
x5 = 6
x6 = 0
x7 = 5/3
x8 = 8
x9 = 3/2
New pattern

Iteration 6

Corresponding dual program

$$\max \begin{bmatrix} 3 & 5 & 8 & 10 & 6 \end{bmatrix} \boldsymbol{\pi}$$

$$\text{s.t.} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{\pi} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Primal program

$$\min \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{x}$$

$$\text{s.t.} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 5 \\ 8 \\ 10 \\ 6 \end{bmatrix}$$

$$\mathbf{x} \geq \mathbf{0}$$

Solution:

$\overline{\mathbf{x}}^6 = \ldots, \overline{z}^6 = \ldots$

Solution:

$\overline{\boldsymbol{\pi}}^6 = \ldots, \overline{v}^6 = \ldots$

When the column generation has converged

- At iteration k, when the reduced cost $\overline{c}^k \geq 0$:
  - The continuous relaxation of the model (second formulation) is solved
  - All patterns needed for an optimal basis have been generated
- Re-insert the integrality restrictions to the current primal program (the restricted master problem, RMP)
- Is the corresponding solution optimal for the original program?
  - Why?
  - Why not?