Stochastic Partial Differential Equations

Stig Larsson

Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg

Computational Mathematical Modelling
A Nordic Graduate Course
Guest lecture, April 27, 2017

Outline

Stochastic heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(u) + g(u)\dot{W}, & x \in \mathcal{D},\ t > 0 \\ u = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0. \end{cases}$$

Stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f(u) + g(u)\dot{W}, & x \in \mathcal{D},\ t > 0 \\ u = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0,\ u_t(0) = u_1. \end{cases}$$

$\dot{W}$ is spatial and temporal noise

Outline

Stochastic Cahn–Hilliard equation (Cahn–Hilliard–Cook):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta v = \dot{W} & \text{in } \mathcal{D} \times [0, T] \\ v = -\Delta u + f(u) & \text{in } \mathcal{D} \times [0, T] \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\mathcal{D} \times [0, T] \\ u(0) = u_0 & \text{in } \mathcal{D} \end{cases}$$

$f(u) = u^3 - u$

Outline

Formulate as an abstract evolution problem in Hilbert space $\mathcal{H}$:

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = F(X)\,\mathrm{d}t + G(X)\,\mathrm{d}W, & t > 0 \\ X(0) = X_0 \end{cases}$$

What does this mean? Strong formulation / variational formulation (depending on how regular X is assumed to be):

$$X(t) = X_0 + \int_0^t (-AX + F(X))\,\mathrm{d}s + \int_0^t G(X)\,\mathrm{d}W$$

Weak formulation:

$$\begin{aligned} \langle X(t), \eta\rangle = \langle X_0, \eta\rangle &+ \int_0^t \langle X(s), -A^*\eta\rangle + \langle F(X(s)), \eta\rangle\,\mathrm{d}s \\ &+ \int_0^t \langle \eta, G(X(s))\,\mathrm{d}W(s)\rangle \quad \forall \eta \in D(A^*) \end{aligned}$$

Outline

We will use the mild formulation:

$$X(t) = \mathrm{e}^{-tA}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A}G(X(s))\,\mathrm{d}W(s)$$

Here $\{\mathrm{e}^{-tA}\}_{t\geq 0}$ is the semigroup of bounded linear operators generated by $-A$.
$\{W(t)\}_{t\geq 0}$ is a Q-Wiener process in another Hilbert space $\mathcal{U}$ and $\int_0^t \cdots \mathrm{d}W$ is a stochastic integral.
We often study the linear case, where $F(X) = f$, $G(X) = B$ are independent of X:

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = f(t)\,\mathrm{d}t + B\,\mathrm{d}W(t), \quad t > 0 \\ X(0) = X_0 \end{cases}$$

Here $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$.
Additive noise: $B\,\mathrm{d}W$. Multiplicative noise: $G(X)\,\mathrm{d}W$.
We shall explain these things.

Notation

- $\mathcal{D} \subset \mathbf{R}^d$ spatial domain, bounded, convex, with polygonal boundary
- $H = L_2(\mathcal{D})$ Lebesgue space
- $\mathcal{H}$, $\mathcal{U}$ real, separable Hilbert spaces
- $\mathcal{L}(\mathcal{U},\mathcal{H})$ bounded linear operators, $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H},\mathcal{H})$

$$\|T\|_{\mathcal{L}(\mathcal{U},\mathcal{H})} = \sup_{u\in\mathcal{U}} \frac{\|Tu\|_{\mathcal{H}}}{\|u\|_{\mathcal{U}}}$$

- $\mathcal{L}_2(\mathcal{U},\mathcal{H})$ Hilbert–Schmidt operators, $\mathrm{HS} = \mathcal{L}_2(\mathcal{H}) = \mathcal{L}_2(\mathcal{H},\mathcal{H})$

$$\|T\|^2_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = \sum_{j=1}^{\infty} \|Te_j\|^2_{\mathcal{H}}, \quad \text{with } \{e_j\}_{j=1}^{\infty} \text{ an arbitrary ON-basis in } \mathcal{U}$$

$$\langle S, T\rangle_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = \sum_{j=1}^{\infty} \langle Se_j, Te_j\rangle_{\mathcal{H}}$$

Notation

- $\mathcal{D} \subset \mathbf{R}^d$ spatial domain, bounded, convex, with polygonal boundary
- $H = L_2(\mathcal{D})$ Lebesgue space
- $\mathcal{H}$, $\mathcal{U}$ real, separable Hilbert spaces
- $\mathcal{L}(\mathcal{U}, \mathcal{H})$ bounded linear operators, $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$

$$\|T\|_{\mathcal{L}(\mathcal{U},\mathcal{H})} = \sup_{u \in \mathcal{U}} \frac{\|Tu\|_{\mathcal{H}}}{\|u\|_{\mathcal{U}}}$$

- $\mathcal{L}_2(\mathcal{U}, \mathcal{H})$ Hilbert–Schmidt operators, $\mathrm{HS} = \mathcal{L}_2(\mathcal{H}) = \mathcal{L}_2(\mathcal{H}, \mathcal{H})$

$$\|T\|^2_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = \sum_{j=1}^{\infty} \|Te_j\|^2_{\mathcal{H}}, \quad \text{with } \{e_j\}_{j=1}^{\infty} \text{ an arbitrary ON-basis in } \mathcal{U}$$

$$\langle S, T \rangle_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = \sum_{j=1}^{\infty} \langle Se_j, Te_j \rangle_{\mathcal{H}}$$

Note:

$$\|ST\|_{\mathcal{L}_2(\mathcal{H})} \leq \|S\|_{\mathcal{L}(\mathcal{H})} \|T\|_{\mathcal{L}_2(\mathcal{H})}$$

Semigroup

A family $\{E(t)\}_{t\geq 0} \subset \mathcal{L}(\mathcal{H})$ is a semigroup of bounded linear operators on $\mathcal{H}$, if

- $E(0) = I, \quad$ (identity operator)
- $E(t+s) = E(t)E(s),\ t,s \geq 0. \quad$ (semigroup property)

It is strongly continuous, or C_0, if

$$\lim_{t\to 0+} E(t)x = x \quad \forall x \in \mathcal{H}.$$

Then the generator of the semigroup is the linear operator G defined by

$$Gx = \lim_{t\to 0+} \frac{E(t)x - x}{t}, \quad D(G) = \{x \in \mathcal{H} : Gx \text{ exists}\}.$$

G is usually unbounded but densely defined and closed.

Semigroup

$u(t) = E(t)u_0$ solves the initial-value problem

$$u'(t) = Gu(t),\ t > 0; \quad u(0) = u_0,$$

if $u_0 \in D(G)$. Therefore, writing $E(t) = \mathrm{e}^{tG}$ is justified.

There are $M \geq 1$, $\omega \in \mathbf{R}$, such that

$$\|E(t)\|_{\mathcal{L}(\mathcal{H})} \leq M\mathrm{e}^{\omega t}, \quad t \geq 0.$$

Without loss of generality we assume $\omega = 0$ (a shift of the operator $G \mapsto G - \omega I$). Contraction semigroup if also $M = 1$.

If $E(t)$ is invertible, $E(t)^{-1} = E(-t)$, then $\{E(t)\}_{t\in\mathbf{R}}$ is a group.

The semigroup is analytic (holomorphic), if $E(t)$ extends to a complex analytic function $E(z)$ in a sector containing the positive real axis $\operatorname{Re} z > 0$. Then the derivative

$$E'(t)u_0 = \frac{\mathrm{d}}{\mathrm{d}t}E(t)u_0 = GE(t)u_0, \quad t > 0,$$

exists for all $u_0 \in \mathcal{H}$, not just for $u_0 \in D(G)$. Moreover,

$$\|E'(t)u_0\|_{\mathcal{H}} = \|GE(t)u_0\|_{\mathcal{H}} \leq Ct^{-1}\|u_0\|_{\mathcal{H}}, \quad t > 0. \tag{1}$$

The inequality (1) is characteristic for analytic semigroups.

Semigroup

On the other hand, we may start with a closed, densely defined, linear operator A and ask for conditions under which $G = -A$ generates a semigroup $E(t) = \mathrm{e}^{-tA}$, so that $u(t) = E(t)u_0$ solves

$$u'(t) + Au(t) = 0, \ t > 0; \quad u(0) = u_0.$$

The non-homogeneous equation

$$u'(t) + Au(t) = f(t), \ t > 0; \quad u(0) = u_0.$$

is then solved by the variation of constants formula (Duhamel's principle):

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(s)\,\mathrm{d}s,$$

provided that f has some small amount of regularity. This is called a mild solution and it is the basis for our semigroup approach to SPDE.

Proof.

Multiply $u'(s) + Au(s) = f(s)$ by the integrating factor $\Phi(s) = E(t-s) = \mathrm{e}^{-(t-s)A}$, $t > s$, and integrate. □

Laplacian

Let $\mathcal{D} \subset \mathbf{R}^d$ be a bounded, convex, polygonal domain. Then

- finite element meshes can be exactly fitted to $\partial\mathcal{D}$;
- we have elliptic regularity:

$$\|v\|_{H^2(\mathcal{D})} \le C\|\Delta v\|_{L_2(\mathcal{D})} \quad \forall v \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}).$$

Here $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplacian. In this way we avoid some technical difficulties associated with the finite element method in smooth domains.

Let $H = L_2(\mathcal{D})$ and $\Lambda = -\Delta$ with $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$. Then Λ is unbounded in H and self-adjoint with compact inverse Λ^{-1}. The spectral theorem gives eigenvalues

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \cdots, \; \lambda_j \to \infty, \; \lambda_j \sim j^{2/d} \text{ as } j \to \infty,$$

and a corresponding orthonormal (ON) basis of eigenvectors $\{\varphi_j\}_{j=1}^\infty$.

Laplacian

Parseval's identity:

$$v = \sum_{j=1}^{\infty} \hat{v}_j \varphi_j, \quad \hat{v}_j = \langle v, \varphi_j \rangle_H, \quad \|v\|_H^2 = \sum_{j=1}^{\infty} \hat{v}_j^2, \quad v \in H.$$

Fractional powers:

$$\Lambda^\alpha v = \sum_{j=1}^{\infty} \lambda_j^\alpha \hat{v}_j \varphi_j, \quad \alpha \in \mathbf{R},$$

$$\|v\|_{\dot{H}^\alpha}^2 = \|\Lambda^{\alpha/2} v\|_H^2 = \sum_{j=1}^{\infty} \lambda_j^\alpha \hat{v}_j^2, \quad \alpha \in \mathbf{R},$$

$$\dot{H}^\alpha = \{v \in H : \|v\|_{\dot{H}^\alpha} < \infty\} = D(\Lambda^{\alpha/2}), \quad \alpha \geq 0,$$

$$\dot{H}^{-\alpha} = \text{ closure of } H \text{ in the } \dot{H}^{-\alpha}\text{-norm}, \quad \alpha > 0,$$

Then $\dot{H}^{-\alpha}$ can be identified with the dual space $(\dot{H}^\alpha)^*$.

Laplacian

The integer order spaces can be identified with standard Sobolev spaces.

Theorem

(i) $\dot{H}^1 = H_0^1(\mathcal{D})$ *with* $\|v\|_{\dot{H}^1} = \|\nabla v\|_{L_2(\mathcal{D})} \simeq \|v\|_{H^1(\mathcal{D})}$ $\forall v \in \dot{H}^1$.
(ii) $\dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ *with* $\|v\|_{\dot{H}^2} = \|\Delta v\|_{L_2(\mathcal{D})} \simeq \|v\|_{H^2(\mathcal{D})}$ $\forall v \in \dot{H}^2$.

Proof.

The proof of (i) is based on the Poincaré inequality and the trace inequality. The proof of (ii) uses also the elliptic regularity. In general, we have only

$$\dot{H}^2 \supset H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}),$$

because, in a nonconvex polygonal domain for example, $\dot{H}^2 = D(\Lambda)$ may contain functions with corner singularities which are not in $H^2(\mathcal{D})$. □

Laplacian

We define the heat semigroup:

$$E(t)v = \mathrm{e}^{-t\Lambda}v = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t}\hat{v}_j\varphi_j.$$

It is analytic in the right half plane $\operatorname{Re} z > 0$. Important bounds:

$$\|E(t)v\|_H \le \|v\|_H, \quad t \ge 0, \tag{2}$$

$$\|D_t^k E(t)v\|_H \le C_k t^{-k}\|v\|_H, \quad t > 0,\ k \ge 0, \tag{3}$$

$$\|\Lambda^\alpha E(t)v\|_H \le C_\alpha t^{-\alpha}\|v\|_H, \quad t > 0,\ \alpha \ge 0, \tag{4}$$

$$\int_0^t \|\Lambda^{1/2}E(s)v\|_H^2\,\mathrm{d}s \le \tfrac{1}{2}\|v\|_H^2, \quad t \ge 0. \tag{5}$$

Recall from (1) that (3) is characteristic for analytic semigroups; and so is (5). They mean that the operator $E(t)$ has a smoothing effect. The smoothing effect in (5) is true for the heat semigroup, but not for analytic semigroups in general.

Laplacian

Proof.

We use Parseval and $x^\alpha e^{-x} \le C_\alpha$ for $x \ge 0$:

$$\|\Lambda^\alpha E(t)v\|_H^2 = \sum_{j=1}^\infty \left(\lambda_j^\alpha e^{-\lambda_j t}\hat{v}_j\right)^2 = t^{-2\alpha}\sum_{j=1}^\infty (\lambda_j t)^{2\alpha} e^{-2\lambda_j t}\hat{v}_j^2$$

$$\le C_\alpha^2 t^{-2\alpha}\sum_{j=1}^\infty \hat{v}_j^2 = C_\alpha^2 t^{-2\alpha}\|v\|_H^2.$$

This proves (2) and (4). Similarly, for (5),

$$\int_0^t \|\Lambda^{1/2}E(s)v\|_H^2\,ds = \int_0^t \sum_{j=1}^\infty \lambda_j e^{-2\lambda_j s}\hat{v}_j^2\,ds$$

$$= \sum_{j=1}^\infty \int_0^t \lambda_j e^{-2\lambda_j s}\,ds\,\hat{v}_j^2 \le \tfrac{1}{2}\|v\|_H^2.$$

□

Laplacian

Remark. The above development based on the spectral representation of fractional powers and the heat semigroup carries over verbatim to more general self-adjoint elliptic operators:

$$\Lambda v = -\nabla \cdot (a(x)\nabla v) + c(x)v \quad \text{with } 0 < a_0 \leq a(x) \leq a_1,\ c(x) \geq 0,$$

for then we still have an ON basis of eigenvectors. For non-self-adjoint elliptic operators, the fractional powers and the semigroup may be constructed by means of an operator calculus based complex contour integration using the resolvent. The bounds (2) and (4) are part of the general theory and (5) can be proved by an energy argument if the operator satisfies the conditions of the Lax–Milgram lemma, for example,

$$\Lambda v = -\nabla \cdot (a(x)\nabla v) + b(x) \cdot \nabla v + c(x)v \quad \text{with } c(x) - \tfrac{1}{2}\nabla \cdot b(x) \geq 0,$$

so that

$$\langle \Lambda v, v \rangle_H \geq c\|v\|^2_{\dot{H}^1}.$$

See the following exercises.

Laplacian

Exercise 1. Prove (5) by the energy method: multiply

$$u'(t) + \Lambda u(t) = 0 \tag{6}$$

by $u(t)$ and integrate.

Exercise 2. Prove the special case $\alpha = \frac{1}{2}$ of (4) by the energy method: multiply (6) by $tu'(t)$ and integrate.

Random variable

Let $\mathcal{U}$ be a separable real Hilbert space and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. A random variable is a measurable mapping $f : \Omega \to \mathcal{U}$, i.e.,

$$f^{-1}(A) \in \mathcal{F} \quad \forall A \in \mathcal{B}(\mathcal{U}) \text{ (= the Borel sigma algebra in } \mathcal{U}).$$

We define Lebesgue-Bochner spaces $L_p(\Omega, \mathcal{U})$:

$$\|f\|_{L_p(\Omega,\mathcal{U})} = \Big(\int_\Omega \|f(\omega)\|_{\mathcal{U}}^p \, \mathrm{d}\mathbf{P}(\omega) \Big)^{1/p} = (\mathbf{E}[\|f\|_{\mathcal{U}}^p])^{1/p},$$

and the expected value

$$\mathbf{E}[f] = \int_\Omega f \, \mathrm{d}\mathbf{P}, \quad f \in L_1(\Omega, \mathcal{U}).$$

Filtration: $\{\mathcal{F}_t\}_{t \geq 0} \subset \mathcal{F}$ increasing family of sigma algebras, $\mathcal{F}_t \subset \mathcal{F}_s$ if $t \leq s$.
Stochastic process: $f = \{f(t)\}_{t \geq 0}$ such that each $f(t)$ is a random variable. It is adapted if $f(t)$ is $\mathcal{F}_t$-measurable.

Brownian motion

Probability space $(\Omega, \mathcal{F}, \mathbf{P})$.
Brownian motion: Real-valued stochastic process $\beta = (\beta(t))_{t \geq 0}$ such that

- $\beta(0) = 0$.
- continuous paths $t \mapsto \beta(t)$ for almost every $\omega \in \Omega$.
- independent increments: $\beta(t) - \beta(s)$ is independent of $\beta(r)$ for $0 \leq r \leq s \leq t$.
- Gaussian law: $\mathbf{P} \circ (\beta(t) - \beta(s))^{-1} \sim \mathcal{N}(0, t - s), \quad s \leq t$. In particular, $\mathbf{E}(\beta(t) - \beta(s)) = 0$, $\mathbf{E}(\beta(t) - \beta(s))^2 = t - s$.

Brownian motion

Probability space $(\Omega, \mathcal{F}, \mathbf{P})$.
Brownian motion: Real-valued stochastic process $\beta = (\beta(t))_{t \geq 0}$ such that

- $\beta(0) = 0$.
- continuous paths $t \mapsto \beta(t)$ for almost every $\omega \in \Omega$.
- independent increments: $\beta(t) - \beta(s)$ is independent of $\beta(r)$ for $0 \leq r \leq s \leq t$.
- Gaussian law: $\mathbf{P} \circ (\beta(t) - \beta(s))^{-1} \sim \mathcal{N}(0, t - s), \quad s \leq t$. In particular, $\mathbf{E}(\beta(t) - \beta(s)) = 0$, $\mathbf{E}(\beta(t) - \beta(s))^2 = t - s$.

It is continuous, but nowhere differentiable. Nevertheless, the Itô integral

$$I = \int_0^T f(t)\,\mathrm{d}\beta(t) = \lim \sum_{j=1}^{N} f(t_{j-1})(\beta(t_j) - \beta(t_{j-1}))$$

can be defined, if the stochastic process f satisfies certain assumptions, and the limit is taken in the correct way...
It is a random variable: $I(\omega) = (\int_0^T f(t)\,\mathrm{d}\beta(t))(\omega)$. It is not path-wise defined: $I(\omega) \neq \int_0^T f(t,\omega)\,\mathrm{d}\beta(t,\omega)$.

Stochastic ODE

$$\begin{cases} \mathrm{d}X(t) = \mu(X(t), t)\,\mathrm{d}t + \sigma(X(t), t)\,\mathrm{d}B(t), \quad t \in [0, T] \\ X(0) = X_0. \end{cases}$$

This means

$$X(t) = X_0 + \int_0^t \mu(X(s), s)\,\mathrm{d}s + \int_0^t \sigma(X(s), s)\,\mathrm{d}B(s), \quad t \in [0, T].$$

Stochastic ODE

$$\begin{cases} \mathrm{d}X(t) = \mu(X(t), t)\,\mathrm{d}t + \sigma(X(t), t)\,\mathrm{d}B(t), & t \in [0, T] \\ X(0) = X_0. \end{cases}$$

This means

$$X(t) = X_0 + \int_0^t \mu(X(s), s)\,\mathrm{d}s + \int_0^t \sigma(X(s), s)\,\mathrm{d}B(s), \quad t \in [0, T].$$

Could be a system:

$$\mathrm{d}X_i = \mu_i(X_1, \ldots, X_n, t)\,\mathrm{d}t + \sum_{j=1}^{m} \sigma_{ij}(X_1, \ldots, X_n, t)\,\mathrm{d}\beta_j(t), \quad i = 1, \ldots, n,$$

$$X = (X_1, \ldots, X_n)^T \in \mathbf{R}^n, \quad \mu : \mathbf{R}^n \times [0, T] \to \mathbf{R}^n, \quad \sigma : \mathbf{R}^n \times [0, T] \to \mathbf{R}^{n \times m},$$

and $B = (\beta_1, \ldots, \beta_m)^T$ an m-dimensional Brownian motion, consisting of m independent Brownian motions β_j.

Covariance

If σ is a constant matrix:

$\mathrm{d}X(t) = \mu(X(t), t)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t)$

The covariance of the noise term is:
(with increments $\Delta B = B(t + \Delta t) - B(t)$)

$$
\begin{aligned}
\mathbf{E}\left[(\sigma\Delta B) \otimes (\sigma\Delta B)\right] &= \mathbf{E}\left[(\sigma\Delta B)(\sigma\Delta B)^T\right] \\
&= \mathbf{E}\left[\sigma\Delta B\Delta B^T\sigma^T\right] \\
&= \sigma\mathbf{E}\left[\Delta B\Delta B^T\right]\sigma^T \\
&= \sigma\,(\Delta t\, I)\,\sigma^T = \Delta t\,\sigma\sigma^T = \Delta t\, Q
\end{aligned}
$$

Covariance matrix: $Q = \sigma\sigma^T \qquad (n \times m) \times (m \times n) = n \times n$

It is symmetric positive semidefinite.

So $\{\sigma B(t)\}_{t \geq 0}$ is a vector-valued Wiener process with covariance matrix $Q = \sigma\sigma^T$.

Covariance

If σ is a constant matrix:

$\mathrm{d}X(t) = \mu(X(t), t)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t)$

The covariance of the noise term is:
(with increments $\Delta B = B(t + \Delta t) - B(t)$)

$$\begin{aligned}
\mathbf{E}\big[(\sigma\Delta B) \otimes (\sigma\Delta B)\big] &= \mathbf{E}\big[(\sigma\Delta B)(\sigma\Delta B)^T\big] \\
&= \mathbf{E}\big[\sigma\Delta B\Delta B^T\sigma^T\big] \\
&= \sigma\mathbf{E}\big[\Delta B\Delta B^T\big]\sigma^T \\
&= \sigma\,(\Delta t\, I)\,\sigma^T = \Delta t\,\sigma\sigma^T = \Delta t\, Q
\end{aligned}$$

Covariance matrix: $Q = \sigma\sigma^T \qquad (n \times m) \times (m \times n) = n \times n$

It is symmetric positive semidefinite.

So $\{\sigma B(t)\}_{t\geq 0}$ is a vector-valued Wiener process with covariance matrix $Q = \sigma\sigma^T$.

Conversely, given Q we may take $\sigma = Q^{1/2}$ and use $Q^{1/2}\,\mathrm{d}B(t)$.

We want to do this in Hilbert space.

Q-Wiener process

We start with a covariance operator $Q \in \mathcal{L}(\mathcal{U})$, self-adjoint, positive semidefinite. We assume that it has an eigenbasis:

$$Qe_j = \gamma_j e_j, \quad \gamma_j \geq 0, \quad \{e_j\}_{j=1}^{\infty} \text{ ON basis in } \mathcal{U}.$$

Let $\beta_j(t)$ be independent identically distributed, real-valued, Brownian motions. Define

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j.$$

Q-Wiener process

We start with a covariance operator $Q \in \mathcal{L}(\mathcal{U})$, self-adjoint, positive semidefinite. We assume that it has an eigenbasis:

$$Qe_j = \gamma_j e_j, \quad \gamma_j \geq 0, \quad \{e_j\}_{j=1}^{\infty} \text{ ON basis in } \mathcal{U}.$$

Let $\beta_j(t)$ be independent identically distributed, real-valued, Brownian motions. Define

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j.$$

Important: how fast $\gamma_j \to 0$. Two important cases:

- $\mathrm{Tr}(Q) < \infty$. $W(t)$ converges in $L_2(\Omega, \mathcal{U})$:

$$\mathbf{E}\Big\| \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j \Big\|_{\mathcal{U}}^2 = \sum_{j=1}^{\infty} \gamma_j \mathbf{E}\big(\beta_j(t)^2\big) = t \sum_{j=1}^{\infty} \gamma_j = t\,\mathrm{Tr}(Q) < \infty$$

Q-Wiener process

We start with a covariance operator $Q \in \mathcal{L}(\mathcal{U})$, self-adjoint, positive semidefinite. We assume that it has an eigenbasis:

$$Qe_j = \gamma_j e_j, \quad \gamma_j \geq 0, \quad \{e_j\}_{j=1}^{\infty} \text{ ON basis in } \mathcal{U}.$$

Let $\beta_j(t)$ be independent identically distributed, real-valued, Brownian motions. Define

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j.$$

Important: how fast $\gamma_j \to 0$. Two important cases:

- $\mathrm{Tr}(Q) < \infty$. $W(t)$ converges in $L_2(\Omega, \mathcal{U})$:
  $$\mathbf{E}\Big\|\sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j\Big\|_{\mathcal{U}}^2 = \sum_{j=1}^{\infty} \gamma_j \mathbf{E}\big(\beta_j(t)^2\big) = t \sum_{j=1}^{\infty} \gamma_j = t\,\mathrm{Tr}(Q) < \infty$$
- $Q = I$, "white noise". $W(t)$ is not $\mathcal{U}$-valued, since $\mathrm{Tr}(I) = \infty$, but converges in a weaker sense; i.e., in a larger space $\mathcal{U}_1$.

Q-Wiener process

If $\mathrm{Tr}(Q) < \infty$:

- $W(0) = 0$.
- continuous paths $t \mapsto W(t)$ in $\mathcal{U}$.
- independent increments: $W(t) - W(s)$ is independent of $W(r)$ for $0 \le r \le s \le t$.
- Gaussian law: $\mathbf{P} \circ (W(t) - W(s))^{-1} \sim \mathcal{N}(0, (t-s)Q), \quad s \le t$

Q-Wiener process

Proof.

(Covariance.) Let $\Delta W = W(t) - W(s)$. Then

$$\begin{aligned}
\big\langle \mathbf{E}\big[\Delta W \otimes \Delta W\big]u, v\big\rangle_{\mathcal{U}} &= \mathbf{E}\big[\langle \Delta W, u\rangle_{\mathcal{U}} \langle \Delta W, v\rangle_{\mathcal{U}}\big] \\
&= \mathbf{E}\Big[\Big\langle \sum_{j=1}^{\infty} \gamma_j^{1/2} \Delta\beta_j e_j, u\Big\rangle_{\mathcal{U}} \Big\langle \sum_{k=1}^{\infty} \gamma_k^{1/2} \Delta\beta_k e_k, v\Big\rangle_{\mathcal{U}}\Big] \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \gamma_j^{1/2} \gamma_k^{1/2} \mathbf{E}\big[\Delta\beta_j \Delta\beta_k\big] \langle e_j, u\rangle_{\mathcal{U}} \langle e_k, v\rangle_{\mathcal{U}} \\
&= (t-s) \sum_{j=1}^{\infty} \gamma_j \langle e_j, u\rangle_{\mathcal{U}} \langle e_j, v\rangle_{\mathcal{U}} = (t-s)\langle Qu, v\rangle_{\mathcal{U}},
\end{aligned}$$

because, by independence,

$$\mathbf{E}\big[\Delta\beta_j \Delta\beta_k\big] = \begin{cases} \mathbf{E}\big[\Delta\beta_j^2\big] = (t-s), & j = k, \\ \mathbf{E}\big[\Delta\beta_j\big]\,\mathbf{E}\big[\Delta\beta_k\big] = 0, & j \neq k. \end{cases}$$

□

Q-Wiener process

Why Hilbert–Schmidt? Let $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ and calculate the norm

$$
\begin{aligned}
&\|B(W(t) - W(s))\|^2_{L_2(\Omega,\mathcal{H})} = \mathbf{E}\big[\|B\Delta W\|^2_{\mathcal{H}}\big] \\
&= \mathbf{E}\Big[\Big\langle \sum_{j=1}^{\infty} \gamma_j^{1/2} \Delta\beta_j Be_j, \sum_{k=1}^{\infty} \gamma_k^{1/2} \Delta\beta_k Be_k \Big\rangle_{\mathcal{H}}\Big] \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \gamma_j^{1/2} \gamma_k^{1/2} \mathbf{E}\big[\Delta\beta_j \Delta\beta_k\big] \langle Be_j, Be_k \rangle_{\mathcal{H}} = (t-s) \sum_{j=1}^{\infty} \gamma_j \|Be_j\|^2_{\mathcal{H}} \\
&= (t-s) \sum_{j=1}^{\infty} \|B\gamma_j^{1/2} e_j\|^2_{\mathcal{H}} = (t-s) \sum_{j=1}^{\infty} \|BQ^{1/2} e_j\|^2_{\mathcal{H}} \\
&= (t-s) \|BQ^{1/2}\|^2_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = (t-s) \|B\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}.
\end{aligned}
$$

Here we used the Hilbert–Schmidt norm of a linear operator $T \colon \mathcal{U} \to \mathcal{H}$:

$$
\|T\|^2_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = \sum_{j=1}^{\infty} \|T\phi_j\|^2_{\mathcal{H}}, \quad \text{arbitrary ON-basis } \{\phi_j\}_{j=1}^{\infty} \text{ in } \mathcal{U}.
$$

Also, it is useful to introduce $\|T\|_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})} = \|TQ^{1/2}\|_{\mathcal{L}_2(\mathcal{U},\mathcal{H})}$.

Wiener integral

We want to define $\int_0^T \Phi(t)\,\mathrm{d}W(t)$, where $\Phi \in L_2([0,T], \mathcal{L}_2^0(\mathcal{U},\mathcal{H}))$ is a deterministic integrand. The construction goes in three steps.

1. Simple functions.

$$0 = t_0 < \cdots < t_j < \cdots < t_N = T,\ \Phi = \sum_{j=0}^{N-1} \Phi_j \mathbf{1}_{[t_j,t_{j+1})},\ \Phi_j \in \mathcal{L}_2^0(\mathcal{U},\mathcal{H}).$$

Define

$$\int_0^T \Phi(t)\,\mathrm{d}W(t) = \sum_{j=0}^{N-1} \Phi_j\big(W(t_{j+1}) - W(t_j)\big).$$

Wiener integral

2. Itô isometry for simple functions. Using the independence of increments and the previous norm calculation:

$$\Big\|\int_0^T \Phi(t)\,\mathrm{d}W(t)\Big\|_{L_2(\Omega,\mathcal{H})}^2 = \mathbf{E}\Big[\Big\|\sum_{j=0}^{N-1}\Phi_j\big(W(t_{j+1})-W(t_j)\big)\Big\|_{\mathcal{H}}^2\Big]$$

$$= \sum_{j=0}^{N-1}\mathbf{E}\Big[\Big\|\Phi_j\big(W(t_{j+1})-W(t_j)\big)\Big\|_{\mathcal{H}}^2\Big]$$

$$= \sum_{j=0}^{N-1}\|\Phi_j\|_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}^2(t_{j+1}-t_j) = \int_0^T \|\Phi(t)\|_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}^2\,\mathrm{d}t.$$

So we have an isometry for simple functions:

$$\Phi \mapsto \int_0^T \Phi\,\mathrm{d}W,$$

$$L_2([0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H})) \to L_2(\Omega,\mathcal{H}).$$

3. Extend to all of $L_2([0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H}))$ by density.

Itô integral

For a random integrand the Itô integral $\int_0^T \Phi \, \mathrm{d}W$ can be defined together with the isometry

$$\mathbf{E}\Big[\Big\|\int_0^T \Phi(t)\,\mathrm{d}W(t)\Big\|_{\mathcal{H}}^2\Big] = \mathbf{E}\Big[\int_0^T \|\Phi(t)\|_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}^2\,\mathrm{d}t\Big]$$

or

$$\Big\|\int_0^T \Phi\,\mathrm{d}W\Big\|_{L_2(\Omega,\mathcal{H})} = \|\Phi\|_{L_2(\Omega\times[0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H}))}$$

Here the process $\Phi\colon [0,T] \to \mathcal{L}_2^0(\mathcal{U},\mathcal{H})$ must be predictable and adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by W and

$$\|\Phi\|_{L_2(\Omega\times[0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H}))}^2 = \mathbf{E}\Big[\int_0^T \|\Phi(t)\|_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}^2\,\mathrm{d}t\Big] < \infty.$$

Recall $\|B\|_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})} = \|BQ^{1/2}\|_{\mathcal{L}_2(\mathcal{U},\mathcal{H})}$.

No details here...

Stochastic evolution equation

Abstract evolution problem in Hilbert space $\mathcal{H}$:

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = F(X)\,\mathrm{d}t + G(X)\,\mathrm{d}W, \quad t > 0 \\ X(0) = X_0 \end{cases}$$

It is now possible to study the mild form of the stochastic evolution equation:

$$\begin{aligned} X(t) = E(t)X_0 &+ \int_0^t E(t-s)F(X(s))\,\mathrm{d}s \\ &+ \int_0^t E(t-s)G(X(s))\,\mathrm{d}W(s), \quad t \geq 0 \end{aligned}$$

where $E(t) = \mathrm{e}^{-tA}$ is a semigroup.

We specialize to the heat and wave equations.

Linear stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t)=0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0)=u_0, & \xi \in \mathcal{D} \end{cases}$$

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = B\,\mathrm{d}W, & t>0 \\ X(0)=X_0 \end{cases}$$

- $\mathcal{H} = \mathcal{U} = H = L_2(\mathcal{D})$, $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, $\mathcal{D} \subset \mathbf{R}^d$, bounded domain
- $A = \Lambda = -\Delta$, $D(\Lambda) = H^2(\mathcal{D}) \cap H^1_0(\mathcal{D})$, $B = I$
- probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- $W(t)$, Q-Wiener process on $\mathcal{U} = H$
- $X(t)$, H-valued stochastic process
- $E(t) = e^{-t\Lambda}$, analytic semigroup generated by $-\Lambda$

Mild solution (stochastic convolution):

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

Regularity

$$\|v\|_{\dot{H}^\beta} = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^\beta \langle v, \phi_j \rangle^2 \Big)^{1/2}, \quad \dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \beta \in \mathbf{R}$$

Mean square norm: $\|v\|^2_{L_2(\Omega,\dot{H}^\beta)} = \mathbf{E}(\|v\|^2_{\dot{H}^\beta}), \quad \beta \in \mathbf{R}$

Hilbert–Schmidt norm:

$$\|T\|^2_{\mathrm{HS}} = \|T\|^2_{\mathcal{L}_2(H,H)} = \sum_{j=1}^{\infty} \|Te_j\|^2_H, \quad \text{any ON basis } \{e_j\}_{j=1}^{\infty}$$

Theorem. If $\|\Lambda^{(\beta-1)/2}\|_{\mathcal{L}_2^0(H)} = \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then $\|X(t)\|_{L_2(\Omega,\dot{H}^\beta)} \leq C\Big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}\Big)$

Regularity

$$\|v\|_{\dot H^\beta} = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^\beta \langle v, \phi_j\rangle^2\Big)^{1/2}, \quad \dot H^\beta = D(\Lambda^{\beta/2}), \quad \beta \in \mathbf{R}$$

Mean square norm: $\|v\|^2_{L_2(\Omega,\dot H^\beta)} = \mathbf{E}(\|v\|^2_{\dot H^\beta}), \quad \beta \in \mathbf{R}$

Hilbert–Schmidt norm:

$$\|T\|^2_{\mathrm{HS}} = \|T\|^2_{\mathcal{L}_2(H,H)} = \sum_{j=1}^{\infty} \|Te_j\|_H^2, \quad \text{any ON basis } \{e_j\}_{j=1}^\infty$$

Theorem. If $\|\Lambda^{(\beta-1)/2}\|_{\mathcal{L}_2^0(H)} = \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then $\|X(t)\|_{L_2(\Omega,\dot H^\beta)} \leq C\Big(\|X_0\|_{L_2(\Omega,\dot H^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}\Big)$

Depends on how fast $\gamma_j \to 0$. Two interesting cases:

- If $\|Q^{1/2}\|^2_{\mathrm{HS}} = \sum_{j=1}^\infty \|Q^{1/2} e_j\|^2 = \sum_{j=1}^\infty \gamma_j = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial \xi^2}$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ for $\beta < 1/2$.

  This is because $\lambda_j \sim j^{2/d}$, so that $\|\Lambda^{(\beta-1)/2}\|^2_{\mathrm{HS}} = \sum_j \lambda_j^{-(1-\beta)} \approx \sum_j j^{-(1-\beta)2/d} < \infty$ iff $d = 1$, $\beta < 1/2$

Temporal regularity for the stochastic heat equation

Take $X_0 = 0$ so that (stochastic convolution) $X(t) = \int_0^t E(t-s)\,\mathrm{d}W(s)$.

Theorem

If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \in [0,1]$, *then*

$$\|X(t) - X(s)\|_{L_2(\Omega,H)} \leq C|t-s|^{\frac{\beta}{2}} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}$$

The linear stochastic wave equation

The linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t > 0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t > 0 \\ u(\xi,0) = u_0,\ \dfrac{\partial u}{\partial t}(\xi,0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

The linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t)=0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0)=u_0,\ \dfrac{\partial u}{\partial t}(\xi,0)=u_1, & \xi \in \mathcal{D} \end{cases}$$

$$\Lambda = -\Delta, \quad D(\Lambda) = \dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$$

The linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0,\ \dfrac{\partial u}{\partial t}(\xi,0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

$$\Lambda = -\Delta, \quad D(\Lambda) = \dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$$

$$\dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \|v\|_{\dot{H}^\beta} = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^\infty \lambda_j^\beta \langle v, \varphi_j \rangle^2 \Big)^{1/2}, \quad \beta \in \mathbf{R}$$

The linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0,\ \dfrac{\partial u}{\partial t}(\xi,0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

$$\Lambda = -\Delta, \quad D(\Lambda) = \dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$$

$$\dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \|v\|_{\dot{H}^\beta} = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^\infty \lambda_j^\beta \langle v, \varphi_j\rangle^2\Big)^{1/2}, \quad \beta \in \mathbf{R}$$

The linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi, t) - \Delta u(\xi, t) = \dot{W}(\xi, t), & \xi \in \mathcal{D} \subset \mathbf{R}^d, \ t > 0 \\ u(\xi, t) = 0, & \xi \in \partial\mathcal{D}, \ t > 0 \\ u(\xi, 0) = u_0, \ \dfrac{\partial u}{\partial t}(\xi, 0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

$$\Lambda = -\Delta, \quad D(\Lambda) = \dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$$

$$\dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \|v\|_{\dot{H}^\beta} = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^\beta \langle v, \varphi_j \rangle^2 \Big)^{1/2}, \quad \beta \in \mathbf{R}$$

$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}u_t \end{bmatrix} + \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \mathrm{d}t = \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W,$$

$$X = \begin{bmatrix} u \\ u_t \end{bmatrix}, \ A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{U} = \dot{H}^0 = L_2(\mathcal{D})$$

The linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0,\ \dfrac{\partial u}{\partial t}(\xi,0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

$$\Lambda = -\Delta, \quad D(\Lambda) = \dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$$

$$\dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \|v\|_{\dot{H}^\beta} = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^\infty \lambda_j^\beta \langle v, \varphi_j \rangle^2 \Big)^{1/2}, \quad \beta \in \mathbf{R}$$

$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}u_t \end{bmatrix} + \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \mathrm{d}t = \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W,$$

$$X = \begin{bmatrix} u \\ u_t \end{bmatrix},\ A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{U} = \dot{H}^0 = L_2(\mathcal{D})$$

$$\mathcal{H} = \mathcal{H}^0 = \dot{H}^0 \times \dot{H}^{-1}, \quad \mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}, \quad D(A) = \mathcal{H}^1$$

Abstract framework

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), & t > 0 \\ X(0) = X_0 \end{cases}$$

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), & t > 0 \\ X(0) = X_0 \end{cases}$$

- $\{X(t)\}_{t\geq 0}$, $\mathcal{H} = \dot{H}^0 \times \dot{H}^{-1}$-valued stochastic process

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \quad t > 0 \\ X(0) = X_0 \end{cases}$$

- $\{X(t)\}_{t\geq 0}$, $\mathcal{H} = \dot{H}^0 \times \dot{H}^{-1}$-valued stochastic process
- $\{W(t)\}_{t\geq 0}$, $\mathcal{U} = \dot{H}^0$-valued Q-Wiener process w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \quad t > 0 \\ X(0) = X_0 \end{cases}$$

- $\{X(t)\}_{t\geq 0}$, $\mathcal{H} = \dot{H}^0 \times \dot{H}^{-1}$-valued stochastic process
- $\{W(t)\}_{t\geq 0}$, $\mathcal{U} = \dot{H}^0$-valued Q-Wiener process w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$
- $E(t) = e^{-tA} = \begin{bmatrix} \cos(t\Lambda^{1/2}) & \Lambda^{-1/2}\sin(t\Lambda^{1/2}) \\ -\Lambda^{1/2}\sin(t\Lambda^{1/2}) & \cos(t\Lambda^{1/2}) \end{bmatrix}$,
  C_0-semigroup on $\mathcal{H}$ but not analytic (actually a group).

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \quad t > 0 \\ X(0) = X_0 \end{cases}$$

- $\{X(t)\}_{t\geq 0}$, $\mathcal{H} = \dot{H}^0 \times \dot{H}^{-1}$-valued stochastic process
- $\{W(t)\}_{t\geq 0}$, $\mathcal{U} = \dot{H}^0$-valued Q-Wiener process w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$
- $E(t) = e^{-tA} = \begin{bmatrix} \cos(t\Lambda^{1/2}) & \Lambda^{-1/2}\sin(t\Lambda^{1/2}) \\ -\Lambda^{1/2}\sin(t\Lambda^{1/2}) & \cos(t\Lambda^{1/2}) \end{bmatrix}$,
  C_0-semigroup on $\mathcal{H}$ but not analytic (actually a group).

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), & t > 0 \\ X(0) = X_0 \end{cases}$$

- $\{X(t)\}_{t\geq 0}$, $\mathcal{H} = \dot{H}^0 \times \dot{H}^{-1}$-valued stochastic process
- $\{W(t)\}_{t\geq 0}$, $\mathcal{U} = \dot{H}^0$-valued Q-Wiener process w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$
- $E(t) = e^{-tA} = \begin{bmatrix} \cos(t\Lambda^{1/2}) & \Lambda^{-1/2}\sin(t\Lambda^{1/2}) \\ -\Lambda^{1/2}\sin(t\Lambda^{1/2}) & \cos(t\Lambda^{1/2}) \end{bmatrix}$,
  C_0-semigroup on $\mathcal{H}$ but not analytic (actually a group).

Here

$$\cos(t\Lambda^{1/2})v = \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})\langle v, \varphi_j\rangle \varphi_j, \quad (\lambda_j, \varphi_j) \text{ are eigenpairs of } \Lambda$$

Regularity

Regularity

Theorem. (With $X(0) = 0$ for simplicity.) If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then there exists a unique mild solution

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \int_0^t E(t-s) B \, \mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda^{-1/2} \sin\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \end{bmatrix}$$

and

$$\|X(t)\|_{L_2(\Omega,\mathcal{H}^\beta)} \leq t \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}. \qquad \mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}$$

Regularity

Theorem. (With $X(0) = 0$ for simplicity.) If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then there exists a unique mild solution

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \int_0^t E(t-s)B\,\mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda^{-1/2}\sin\left((t-s)\Lambda^{1/2}\right)\mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda^{1/2}\right)\mathrm{d}W(s) \end{bmatrix}$$

and

$$\|X(t)\|_{L_2(\Omega,\mathcal{H}^\beta)} \leq t\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}. \qquad \mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}$$

Two cases:

- If $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.

Regularity

Theorem. (With $X(0) = 0$ for simplicity.) If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then there exists a unique mild solution

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \int_0^t E(t-s) B \, \mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda^{-1/2} \sin\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \end{bmatrix}$$

and

$$\|X(t)\|_{L_2(\Omega, \mathcal{H}^\beta)} \leq t \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}. \qquad \mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}$$

Two cases:

- If $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.
- If $Q = I$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ iff $d = 1$, $\beta < 1/2$.

Recall the linear stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subset \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0, & \xi \in \mathcal{D} \end{cases}$$

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = \mathrm{d}W, & t>0 \\ X(0) = X_0 \end{cases}$$

- $\mathcal{H} = \mathcal{U} = H = L_2(\mathcal{D})$, $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, $\mathcal{D} \subset \mathbf{R}^d$, bounded domain
- $A = \Lambda = -\Delta$, $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, $B = I$
- probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- $W(t)$, Q-Wiener process on $\mathcal{U} = H$
- $X(t)$, $\mathcal{H}$-valued stochastic process
- $E(t) = e^{-t\Lambda}$, analytic semigroup generated by $-\Lambda$

Mild solution (stochastic convolution):

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

The finite element method

- family of triangulations $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h
- finite element spaces $\{S_h\}_{0<h<1}$, $\quad S_h \subset H_0^1(\mathcal{D}) = \dot{H}^1$
- S_h continuous piecewise linear functions
- $X_h(t) \in S_h$; $\langle \mathrm{d}X_h, \chi \rangle + \langle \nabla X_h, \nabla \chi \rangle \,\mathrm{d}t = \langle \mathrm{d}W, \chi \rangle \ \forall \chi \in S_h,\ t > 0$
- $\Lambda_h \colon S_h \to S_h,$ discrete Laplacian, $\langle \Lambda_h \psi, \chi \rangle = \langle \nabla \psi, \nabla \chi \rangle \ \forall \psi, \chi \in S_h$
- $A_h = \Lambda_h$
- $P_h \colon L_2 \to S_h,$ orthogonal projection, $\quad \langle P_h f, \chi \rangle = \langle f, \chi \rangle \ \forall \chi \in S_h$

$$\begin{cases} X_h(t) \in S_h, \quad X_h(0) = P_h X_0 \\ \mathrm{d}X_h + \Lambda_h X_h \,\mathrm{d}t = P_h \,\mathrm{d}W, \quad t > 0 \end{cases}$$

$P_h W(t)$ is a Q_h-Wiener process with $Q_h = P_h Q P_h$.

Mild solution, with $E_h(t) v_h = e^{-t\Lambda_h} v_h = \sum_{j=1}^{N_h} \mathrm{e}^{-t\lambda_{h,j}} \langle v_h, \varphi_{h,j} \rangle \varphi_{h,j}$:

$$X_h(t) = E_h(t) P_h X_0 + \int_0^t E_h(t-s) P_h \,\mathrm{d}W(s)$$

Strong convergence

Theorem

If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \in [0,2]$*, then*

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \leq Ch^\beta \Big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} \Big).$$

Optimal result: the order of regularity equals the order of convergence.

Two cases:

- If $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then the convergence rate is $O(h)$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial \xi^2}$, then the rate is almost $O(h^{1/2})$.

No result for $Q = I$, $d \geq 2$.

Time discretization

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = \mathrm{d}W, & t > 0 \\ X(0) = X_0 \end{cases}$$

The implicit Euler method (implicit Euler–Maruyama method):
$k = \Delta t,\ t_n = nk,\ \Delta W^n = W(t_n) - W(t_{n-1})$

$$\begin{cases} X_h^n \in S_h, \quad X_h^0 = P_h X_0 \\ X_h^n - X_h^{n-1} + kA_h X_h^n = P_h \Delta W^n, \end{cases}$$

$$X_h^n = E_{kh} X_h^{n-1} + E_{kh} P_h \Delta W^n, \quad E_{kh} = (I + kA_h)^{-1}$$

$$X_h^n = E_{kh}^n P_h X_0 + \sum_{j=1}^{n} E_{kh}^{n-j+1} P_h \Delta W^j$$

$$X(t_n) = E(t_n) X_0 + \int_0^{t_n} E(t_n - s)\,\mathrm{d}W(s)$$

Strong convergence

Theorem

If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \in [0,2]$*, then, with* $e^n = X_h^n - X(t_n)$*,*

$$\|e^n\|_{L_2(\Omega,H)} \le C(k^{\beta/2} + h^{\beta})\Big(\|X_0\|_{L_2(\Omega,\dot{H}^{\beta})} + \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}\Big)$$

The reason why we can have k^1 (when $\beta = 2$) is that the Euler–Maruyama method is exact in the stochastic integral for additive noise. For multiplicative noise we get at most $k^{1/2}$.

Implementation

Euler's method for the stochastic heat equation

$$\begin{cases} X^n \in S_h, \quad X^0 = P_h u_0 \\ X^n - X^{n-1} + \Delta t \Lambda_h X^n = P_h \Delta W^n \end{cases}$$

$$(X^n - X^{n-1}, \chi) + \Delta t(\nabla X^n, \nabla \chi) = (\underbrace{\Delta W^n}_{\in L_2(\Omega, \dot{H}^{-1})}, \chi), \quad \forall \chi \in S_h$$

$$X^n(x) = \sum_{k=1}^{N_h} X_k^n \phi_k(x), \quad \chi = \phi_j, \quad \{\phi_j\}_1^{N_h} \text{ finite element basis functions}$$

$$\sum_{k=1}^{N_h} X_k^n(\phi_k, \phi_j) + \Delta t \sum_{k=1}^{N_h} X_k^n(\nabla \phi_k, \nabla \phi_j) = \sum_{k=1}^{N_h} X_k^{n-1}(\phi_k, \phi_j) + (\Delta W^n, \phi_j)$$

$$\mathbf{M}\mathbf{X}^n + \Delta t \mathbf{K}\mathbf{X}^n = \mathbf{M}\mathbf{X}^{n-1} + \mathbf{b}^n$$

Implementation

How to simulate $\mathbf{b}_j^n = (\Delta W^n, \phi_j) = (W(t_n) - W(t_{n-1}), \phi_j)$?

Covariance of $\mathbf{b}^n$:

$$\mathbf{E}(\mathbf{b}_i^n \mathbf{b}_j^n) = \mathbf{E}\big((\Delta W^n, \phi_i)(\Delta W^n, \phi_j)\big) = \Delta t (Q\phi_i, \phi_j)$$

In other words:

$$\mathbf{E}(\mathbf{b}^n \otimes \mathbf{b}^n) = \Delta t \mathbf{Q}, \quad \mathbf{Q}_{ij} = (Q\phi_i, \phi_j).$$

This assumes that the action of the covariance operator is known (computable). For example, integral operator with known kernel: $(Qf)(x) = \int_{\mathcal{D}} q(x,y) f(y)\,\mathrm{d}y$.

Cholesky factorization: $\mathbf{Q} = \mathbf{L}\mathbf{L}^T$, expensive, but done only once.

Take $\mathbf{b}^n = \sqrt{\Delta t}\,\mathbf{L}\beta^n$, where $\beta^n \in \mathbf{R}^{N_h}$, $n = 1, 2, \ldots$, are $\mathcal{N}(0, \mathbf{I})$, that is, generate one random vector in each time step, the components are independent normally distributed random numbers.

Then

$$\begin{aligned}\mathbf{E}(\mathbf{b}^n \otimes \mathbf{b}^n) &= \mathbf{E}(\mathbf{b}^n (\mathbf{b}^n)^T) = \Delta t \mathbf{E}(\mathbf{L}\beta^n (\mathbf{L}\beta^n)^T) \\ &= \Delta t \mathbf{L} \mathbf{E}(\beta^n (\beta^n)^T) \mathbf{L}^T = \Delta t \mathbf{L}\mathbf{L}^T = \Delta t \mathbf{Q}\end{aligned}$$

Implementation

One situation where the action of Q is known is $Q = I$. Then $\mathbf{Q}_{ij} = (Q\phi_i, \phi_j) = (\phi_i, \phi_j)$, that is, $\mathbf{Q} = \mathbf{M}$, the mass matrix. It is sparse so the Cholesky factorization is not too expensive. It can also be approximated by the lumped mass matrix $\mathbf{M}_L$, which is diagonal and $\mathbf{M}_L^{1/2}$ is easily computed. Then $\mathbf{b}^n = \sqrt{\Delta t}\,\mathbf{M}_L^{1/2}\beta^n$ can be used.

But $Q = I$ is of no interest unless $d = 1$, as we have seen.

However, it can be used (also for $d \geq 1$) to generate noise increments ΔW with prescribed covariance from the Matérn class of covariance kernels. Let ΔW_I be a noise increment with $Q = I$ and solve the equation

$$(\kappa I - \Delta)^{(\nu+1)/2} \Delta W = \Delta W_I \quad \text{in } \mathcal{D}.$$

Then ΔW will have a covariance from the Matérn class with parameters κ, ν. Its finite element approximation will serve as the vector $\mathbf{b}$ above. But this equation is, in general, of fractional order $\nu + 1$ and it is therefore not straightforward to solve.

Implementation

Another approach: truncate the orthogonal expansion (Karhunen–Loève expansion)

$$W(t) = \sum_{k=1}^{\infty} \gamma_k^{1/2} \beta_k(t) e_k \approx \sum_{k=1}^{M} \gamma_k^{1/2} \beta_k(t) e_k, \quad Qe_k = \gamma_k e_k.$$

The truncated expansion can be inserted in the finite element equation. This assumes that the eigenvectors of Q are known. The eigenvalues can be chosen with the desired rate of convergence $\gamma_k \to 0$.

Random partial differential equation

$$\text{Find } u(\omega) \in V : a(\omega; u(\omega), v) = (f(\omega), v) \quad \forall v \in V.$$

Probability space $(\Omega, \mathcal{F}, \mathbf{P})$.
Gelfand triple $V \subset H \subset V^*$ of Hilbert spaces.
$A \in \mathcal{L}(V, V^*)$, $f \in V^*$, $\quad$ **P**-almost surely.

Bilinear form $a(\omega; u, v) := {}_{V^*}\langle A(\omega)u, v\rangle_V$ such that

$$|a(\omega; u, v)| \leq A_{\max}(\omega)\|u\|_V\|V\|v, \quad u, v \in V,$$
$$a(\omega; v, v) \geq A_{\min}(\omega)\|v\|_V^2, \quad v \in V.$$

with some positive random variables $A_{\max}, A_{\min}$.

Approach: prove bounds ω-wise, then take $L^p(\Omega; \cdots)$-norms.

Basis for analysis of Monte Carlo and Multilevel Monte Carlo methods.

Random PDE

$$A(\omega)v = -\nabla \cdot (a(x,\omega)\nabla v), \quad a(\omega; v, w) = (a(\cdot,\omega)\nabla v, \nabla w)$$

where the diffusion coefficient is a random field:

$$a(x,\omega) = \bar{a}(x) + \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(\omega) e_j(x).$$

Here $\beta_j \sim \mathcal{N}(0,1)$ are independent real random variables and (γ_j, e_j) are the eigenpairs of a covariance operator Q. Similar for $f(x,\omega)$.
The smoothness of a depends on how fast $\gamma_j \to 0$.

Random PDE

$$A(\omega)v = -\nabla \cdot (a(x,\omega)\nabla v), \quad a(\omega; v, w) = (a(\cdot,\omega)\nabla v, \nabla w)$$

where the diffusion coefficient is a random field:

$$a(x,\omega) = \bar{a}(x) + \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(\omega) e_j(x).$$

Here $\beta_j \sim \mathcal{N}(0,1)$ are independent real random variables and (γ_j, e_j) are the eigenpairs of a covariance operator Q. Similar for $f(x,\omega)$.
The smoothness of a depends on how fast $\gamma_j \to 0$.

Difficulty: $a(x,\omega)$ may be < 0 with some small probability.

Random PDE

$$A(\omega)v = -\nabla \cdot (a(x,\omega)\nabla v), \quad a(\omega; v, w) = (a(\cdot,\omega)\nabla v, \nabla w)$$

where the diffusion coefficient is a random field:

$$a(x,\omega) = \bar{a}(x) + \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(\omega) e_j(x).$$

Here $\beta_j \sim \mathcal{N}(0,1)$ are independent real random variables and (γ_j, e_j) are the eigenpairs of a covariance operator Q. Similar for $f(x,\omega)$.
The smoothness of a depends on how fast $\gamma_j \to 0$.

Difficulty: $a(x,\omega)$ may be < 0 with some small probability.

Log-normal random field: $a(x,\omega) = \exp(g(x,\omega))$ where g is as above.

Random PDE

$$A(\omega)v = -\nabla \cdot (a(x,\omega)\nabla v), \quad a(\omega; v, w) = (a(\cdot,\omega)\nabla v, \nabla w)$$

where the diffusion coefficient is a random field:

$$a(x,\omega) = \bar{a}(x) + \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(\omega) e_j(x).$$

Here $\beta_j \sim \mathcal{N}(0,1)$ are independent real random variables and (γ_j, e_j) are the eigenpairs of a covariance operator Q. Similar for $f(x,\omega)$.
The smoothness of a depends on how fast $\gamma_j \to 0$.

Difficulty: $a(x,\omega)$ may be < 0 with some small probability.

Log-normal random field: $a(x,\omega) = \exp(g(x,\omega))$ where g is as above.

Solve for fixed ω, take statistics with respect to ω.

Random PDE

$$A(\omega)v = -\nabla \cdot (a(x,\omega)\nabla v), \quad a(\omega; v, w) = (a(\cdot,\omega)\nabla v, \nabla w)$$

where the diffusion coefficient is a random field:

$$a(x,\omega) = \bar{a}(x) + \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(\omega) e_j(x).$$

Here $\beta_j \sim \mathcal{N}(0,1)$ are independent real random variables and (γ_j, e_j) are the eigenpairs of a covariance operator Q. Similar for $f(x,\omega)$.
The smoothness of a depends on how fast $\gamma_j \to 0$.

Difficulty: $a(x,\omega)$ may be < 0 with some small probability.

Log-normal random field: $a(x,\omega) = \exp(g(x,\omega))$ where g is as above.

Solve for fixed ω, take statistics with respect to ω.
"Uncertainty Quantification"