1 Finite Element Approximation

1.1 Stationary Problems

Problem 1.1 *Consider the two-point boundary value problem*

$$\begin{aligned} -u'' &= f, \quad 0 < x < 1, \\ u(0) &= u(1) = 0. \end{aligned} \tag{1}$$

Let $V = \{v : \|v\| + \|v'\| < \infty,\ v(0) = v(1) = 0\}$.

a. Use V *to derive a variational formulation of* (1).

b. Discuss why V *is a valid as a vector space of test functions.*

c. Classify the following functions as admissible test functions or not.

$$\sin \pi x, \qquad x^2, \qquad x \ln x, \qquad e^x - 1, \qquad x(1-x).$$

Problem 1.2 *Assume that* $u(0) = u(1) = 0$, *and that* u *satisfies*

$$\int_0^1 u'v'\,dx = \int_0^1 fv\,dx,$$

for all $v \in V = \{v : \|v\| + \|v'\| < \infty,\ v(0) = v(1) = 0\}$.

a. Show that u *minimizes the functional*

$$F(v) = \frac{1}{2}\int_0^1 (v')^2\,dx - \int_0^1 fv\,dx.$$

Hint: write $F(v) = F(u+w) = F(u) + \ldots \geq F(u)$.

b. Prove that the above minimization problem is equivalent to

$$\begin{aligned} -u'' &= f, \\ u(0) &= u(1) = 0. \end{aligned}$$

Problem 1.3 *Consider the two-point boundary value problem*

$$\begin{aligned} -u'' &= 1, \quad 0 < x < 1, \\ u(0) &= u(1) = 0. \end{aligned} \tag{3}$$

Let $\mathcal{T}_h : x_j = \frac{j}{4}$*,* $j = 0, 1, \ldots, 4$*, denote a partition of the interval* $0 < x < 1$ *into four subintervals of equal length* $h = 1/4$ *and let* V_h *be the corresponding space of continuous piecewise linear functions vanishing at* $x = 0$ *and* $x = 1$*.*

a. *Compute a finite element approximation* $U \in V_h$ *to* (3)*.*

b. *Prove that* $U \in V_h$ *is unique.*

Problem 1.4 *Consider once again the two-point boundary value problem*

$$\begin{aligned} -u'' &= f, \quad 0 < x < 1, \\ u(0) &= u(1) = 0. \end{aligned}$$

a. *Prove that the finite element approximation* $U \in V_h$ *to* u *satisfies*

$$\|(u - U)'\| \leq \|(u - v)'\|,$$

for all $v \in V_h$*.*

b. *Use this result to deduce that*

$$\|(u - \pi_h u)'\| \leq C\|hu''\|,$$

where C *is a constant and* $\pi_h u$ *a piecewise linear interpolant to* u*.*

Problem 1.5 *Consider the two-point boundary value problem*

$$\begin{aligned} -(au')' &= f, \quad 0 < x < 1, \\ u(0) = 0, &\quad a(1)u'(1) = g_1, \end{aligned} \tag{5}$$

where $a > 0$ *is a positive function and* g_1 *is a constant.*

a. *Derive the variational formulation of* (5)*.*

b. *Discuss how the boundary conditions are implemented.*

Problem 1.6 *Consider the two-point boundary value problem*

$$\begin{aligned} -u'' &= 0, \quad 0 < x < 1, \\ u(0) &= 0, \quad u'(1) = 7. \end{aligned} \tag{6}$$

Divide the interval $0 \le x \le 1$ *into two subintervals of length* $h = \frac{1}{2}$ *and let* V_h *be the corresponding space of continuous piecewise linear functions vanishing at* $x = 0$.

a. Formulate a finite element method for (6).

b. Calculate by hand the finite element approximation $U \in V_h$ *to* (6).

c. Study how the boundary condition at $x = 1$ *is approximated.*

Problem 1.7 *Consider the two-point boundary value problem*

$$\begin{aligned} -u'' &= 0, \quad 0 < x < 1, \\ u'(0) &= 5, \quad u(1) = 0. \end{aligned} \tag{7}$$

Let $\mathcal{T}_h : x_j = jh,\ j = 0, 1, \ldots, N,\ h = 1/N$ *be a uniform partition of the interval* $0 < x < 1$ *into* N *subintervals and let* V_h *be the corresponding space of continuous piecewise linear functions.*

a. Use V_h *to formulate a finite element method for* (7).

b. Compute the finite element approximation $U \in V_h$ *assuming* $N = 3$.

Problem 1.8 *Consider the problem of finding a solution approximation to*

$$\begin{aligned} -u'' &= 1, \quad 0 < x < 1, \\ u'(0) &= u'(1) = 0. \end{aligned} \tag{8}$$

Let $\mathcal{T}_h$ *be a partition of the interval* $0 < x < 1$ *into two subintervals of equal length* $h = \frac{1}{2}$ *and let* V_h *be the corresponding space of continuous piecewise linear functions.*

a. Find the exact solution u *to* (8) *by integrating twice.*

b. Compute a finite element approximation $U \in V_h$ *to* u *if possible.*

c. Explain why this two-point boundary value problem is said to be ill posed.

Problem 1.9 *Consider the two-point boundary value problem*

$$\begin{aligned} -((1+x)u')' = 0, \quad 0 < x < 1, \\ u(0) = 0, \quad u'(1) = 1. \end{aligned} \tag{9}$$

Divide the interval $0 < x < 1$ *into* 3 *subintervals of equal length* $h = \frac{1}{3}$ *and let* V_h *be the corresponding space of continuous piecewise linear functions vanishing at* $x = 0$.

a. *Use* V_h *to formulate a finite element method for* (9).

b. *Verify that the stiffness matrix* $\boldsymbol{A}$ *and load vector* $\boldsymbol{b}$ *are given by*

$$\boldsymbol{A} = \frac{1}{2}\begin{bmatrix} 16 & -9 & 0 \\ -9 & 20 & -11 \\ 0 & -11 & 11 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

c. *Show that* $\boldsymbol{A}$ *is symmetric, tridiagonal, and positive definite.*

d. *Derive a simple way to compute the energy norm* $\|U\|_E^2$, *defined by*

$$\|U\|_E^2 = \int_0^1 (1+x)U'(x)^2\,dx,$$

where $U \in V_h$ *is the finite element solution approximation.*

Problem 1.10 *Consider the two-point boundary value problem*

$$\begin{aligned} -u'' = 0, \quad 0 < x < 1, \\ u(0) = 0, \quad u'(1) = k(u(1) - 1). \end{aligned} \tag{10}$$

Let $\mathcal{T}_h : 0 = x_0 < x_1 < x_2 < x_3 = 1$, *where* $x_1 = \frac{1}{3}$ *and* $x_2 = \frac{2}{3}$ *be a partition of the interval* $0 \le x \le 1$ *and let* V_h *be the corresponding space of continuous piecewise linear functions, which vanish at* $x = 0$.

a. *Compute a solution approximation* $U \in V_h$ *to* (10) *assuming* $k = 1$

b. *Discuss how the parameter* k *influence the boundary condition at* $x = 1$.

Problem 1.11 *Consider the finite element method applied to*

$$-u'' = 0, \quad 0 < x < 1,$$
$$u(0) = \alpha, \quad u'(1) = \beta,$$

where α *and* β *are given constants. Assume that the interval* $0 \le x \le 1$ *is divided into three subintervals of equal length* $h = 1/3$ *and that* $\{\varphi_j\}_0^3$ *is a nodal basis of* V_h*, the corresponding space of continuous piecewise linear functions.*

a. *Verify that the ansatz*

$$U(x) = \alpha\varphi_0(x) + \xi_1\varphi_1(x) + \xi_2\varphi_2(x) + \xi_3\varphi_3(x)$$

yields the following system of equations

$$\frac{1}{h}\begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix}. \tag{11}$$

b. *If* $\alpha = 2$ *and* $\beta = 3$ *show that* (11) *can be reduced to*

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} -2h^{-1} \\ 0 \\ 3 \end{bmatrix}.$$

c. *Solve the above system of equations and find* $U(x)$.

Problem 1.12 *Compute a finite element solution approximation to*

$$-u'' + u = 1, \quad 0 \le x \le 1,$$
$$u(0) = u(1) = 0,$$

using the continuous piecewise linear ansatz $U = \xi_1\varphi_1 + \xi_2\varphi_2$*, where*

$$\varphi_1 = \begin{cases} 3x, & 0 < x < \frac{1}{3} \\ 2 - 3x, & \frac{1}{3} < x < \frac{2}{3} \\ 0, & \frac{2}{3} < x < 1 \end{cases}, \quad \varphi_2 = \begin{cases} 0, & 0 < x < \frac{1}{3} \\ 3x - 1, & \frac{1}{3} < x < \frac{2}{3} \\ 3 - 3x, & \frac{2}{3} < x < 1 \end{cases}.$$

Problem 1.13 *Consider the following eigenvalue problem*

$$-au'' + bu = 0, \quad 0 \le x \le 1,$$
$$u(0) = u'(1) = 0,$$

where a, $b > 0$ *are constants. Let* $\mathcal{T}_h : 0 = x_0 < x_1 < \ldots < x_N = 1$, *be a non-uniform partition of the interval* $0 \le x \le x$ *into* N *intervals of length* $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$ *and let* V_h *be the corresponding space of continuous piecewise linear functions. Compute the stiffness and mass matrices.*