1 Projection Methods

1.1 Linear Least Squares

Problem 1.1 *Use the method of least squares to solve the following systems of linear equations.*

$$a.\ \begin{cases} -x_1 + x_2 = 16 \\ 2x_1 + x_2 = -9 \\ x_1 - 2x_2 = -12 \end{cases} \qquad b.\ \begin{cases} x_1 + x_2 = 3 \\ -2x_1 + 3x_2 = 1 \\ 2x_1 - x_2 = 2 \end{cases}$$

$$c.\ \begin{cases} x_1 + 2x_2 = 3 \\ -2x_1 + x_2 = -4 \\ x_1 - 3x_2 = -2 \\ -x_1 + x_2 = -1 \\ 2x_1 + x_2 = 5 \end{cases} \qquad d.\ \begin{cases} x_1 + x_2 + x_3 = 4 \\ -x_1 + x_2 + x_3 = 0 \\ -x_2 + x_3 = 1 \\ x_1 + x_3 = 2 \end{cases}$$

$$e.\ \begin{cases} x_1 + x_2 + x_3 = 7 \\ x_1 + x_2 - x_3 = -1 \\ x_1 - x_2 + x_3 = 1 \\ x_1 - x_2 - x_3 = 3 \end{cases}$$

Problem 1.2 *Determine the line $y = b + ct$ that fits the following pairs of data (t, y) best.*

a.

t	1	2	3	4	5
y	1	5	2	7	10

b.

t	1	2	3	4	5
y	5	6	10	12	17

c.

t	1	2	3	4	5
y	2	3	1	1	-2

Problem 1.3 *Determine the parameters a and b such that the parabolic curve $y = ax^2 + bx + c$ fits the following values of x and y best in the least squares sense.*

a.

x	-2	-1	0	1	2
y	2	1	1	2	3

b.

x	-1	0	1	2
y	2	2	1	0

Problem 1.4 *Let x be the solution of the least squares problem $Ax \approx b$, where*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

Let $r - b - Ax$ be the corresponding residual. Which of the following three vectors is a possible value for r?

$$a. \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad b. \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \qquad c. \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Problem 1.5 *Set up and solve the linear least squares system $Ax \approx b$ for fitting the model function $f(t, x) = x_1 t + x_2 e^t$ to the three data points $(1, 2)$ $(2, 3)$ and $(3, 5)$.*

Problem 1.6 *True or false: At the solution to a linear least squares problem $Ax \approx b$, the residual vector $r = b - Ax$ is orthogonal to the column space of A.*

1.2 Galerkin's Method

Problem 1.7 *We want to find a solution approximation $U(x)$ to*

$$-u''(x) = 1, \quad 0 < x < 1, \quad u(0) = u(1) = 0,$$

using the ansatz $U(x) = A \sin \pi x + B \sin 2\pi x$.

a. *Calculate the exact solution $u(x)$.*

b. *Write down the residual $R(x) = -U''(x) - 1$*

c. *Use the orthogonality condition*

$$\int_0^1 R(x) \sin \pi n x \, dx = 0, \quad n = 1, 2,$$

to determine the constants A and B.

d. Plot the error $e(x) = u(x) - U(x)$.

Problem 1.8 *Consider the boundary value problem*

$$-u''(x) + u(x) = x, \quad 0 < x < 1, \quad u(0) = u(1) = 0.$$

a. Verify that the exact solution of the problem is given by

$$u(x) = x - \frac{\sinh x}{\sinh 1}.$$

b. Let $U(x)$ *be a solution approximation defined by*

$$U(x) = A \sin \pi x + B \sin 2\pi x + C \sin 3\pi x,$$

where A, B, *and* C *are unknown constants. Compute the residual function*

$$R(x) = -U''(x) + U(x) - x.$$

c. Use the orthogonality condition

$$\int_0^1 R(x) \sin \pi n x \, dx = 0, \quad n = 1, 2, 3,$$

to determine the constants A, B, *and* C.

Problem 1.9 *Let* $U(x) = \xi_0 \phi_0(x) + \xi_1 \phi_1(x)$ *be a solution approximation to*

$$-u''(x) = x - 1, \quad 0 < x < \pi, \quad u'(0) = u(\pi) = 0,$$

where ξ_i, $i = 0, 1$, *are unknown coefficients and*

$$\phi_0(x) = \cos \frac{x}{2}, \qquad \phi_1(x) = \cos \frac{3x}{2}.$$

a. Find the analytical solution $u(x)$.

b. Define the approximate solution residual $R(x)$.

c. Compute the constants ξ_i *using the orthogonality condition*

$$\int_0^1 R(x)\, \phi_i(x) \, dx = 0, \quad i = 0, 1,$$

i.e., by projecting $R(x)$ *onto the vector space spanned by* $\phi_0(x)$ *and* $\phi_1(x)$.

Problem 1.10 *Use the projection technique of the previous exercises to solve*

$$-u''(x) = 0, \quad 0 < x < \pi, \quad u(0) = 0, \quad u(\pi) = 2,$$

assuming that $U(x) = A \sin x + B \sin 2x + C \sin 3x + \frac{2}{\pi^2} x^2$.

1.3 Interpolation Method

Problem 1.11 *Show that*

$$\mathcal{P}^q(a,b) := \{p(x)|p(x) \text{ is a polynomial of degree } \leq q\},$$

is a vector space but

$$P^q(a,b) := \{p(x)|p(x) \text{ is a polynomial of degree } = q\},$$

is not! a vector space.

Problem 1.12 *Compute formulas for the linear interpolant of a continuous function f through the points a and $(b{+}a)/2$. Plot the corresponding Lagrange basis functions.*

Problem 1.13 *Prove the following interpolation error estimate:*

$$||\Pi_1 f - f||_{L_\infty(a,b)} \leq \frac{1}{8}(b-a)^2||f''||_{L_\infty(a,b)}.$$

Problem 1.14 *Compute and graph $\pi_4\Big(e^{-8x^2}\Big)$ on $[-2,2]$, which interpolates e^{-8x^2} at 5 equally spaced points in $[-2,2]$.*

Problem 1.15 *Write down a basis for the set of piecewise quadratic polynomials $W_h^{(2)}$ on a partition $a = x_0 < x_1 < x_2 < \ldots < x_{m+1} = b$ of (a,b) into subintervals $I_i = (x_{i-1}, x_i)$, where*

$$W_h^{(q)} = \{v : v|_{I_i} \in \mathcal{P}^q(I_i), i = 1, \ldots, m+1\}$$

Problem 1.16 *Prove that*

$$\int_{x_0}^{x_1} f'\Big(\frac{x_1+x_0}{2}\Big)\Big(x - \frac{x_1+x_0}{2}\Big)\,dx = 0.$$

Problem 1.17 *Prove that*

$$\begin{aligned}\Big|\int_{x_0}^{x_1} f(x)\,dx - f\Big(\frac{x_1+x_0}{2}\Big)(x-x_0)\Big| \\ &\leq \frac{1}{2}\max_{[x_0,x_1]}|f''|\int_{x_0}^{x_1}\Big(x - \frac{x_1+x_0}{2}\Big)^2 dx \\ &\leq \frac{1}{24}(x_1-x_0)^3\max_{[x_0,x_1]}|f''|.\end{aligned}$$

Answer 1.1

a. $x_1 = -7$, $x_2 = 4$

b. $x_1 = 1.66$, $x_2 = 4.42$

c. $x_1 = 2$, $x_2 = 1$

d. $x_1 = 1.6$, $x_2 = 0.6$, $x_3 = 1.2$

e. $x_1 = 1$, $x_2 = 1$, $x_3 = 3$

Answer 1.2

a. $y = 2t - 1$

b. $y = 3t + 1$

c. $y = 4 - t$

Answer 1.3

a. $y = \frac{1}{70}(25x^2 + 21x + 76)$

b. $y = \frac{1}{20}(-5x^2 - 9x + 37)$

Answer 1.4 c. because r must be orthogonal against all columns of A.

Answer 1.5 $x_1 = 1.5942$, $x_2 = 0.0088$

Answer 1.6 Yes

Answer 1.7

a. $u(x) = \frac{1}{2}x(1 - x)$

b. $R(x) = \pi^2 A \sin \pi x + 4\pi^2 B \sin 2\pi x - 1$

c. $A = 4/\pi^3$ and $B = 0$

d. -

Answer 1.8

a. -

b. $R(x) = (\pi^2+1)A\sin\pi x + (4\pi^2+1)B\sin 2\pi x + (9\pi^2+1)C\sin 3\pi x - x$

c. $A = \dfrac{2}{\pi(\pi^2+1)}$, $B = -\dfrac{1}{\pi(4\pi^2+1)}$ and $C = \dfrac{2}{3\pi(9\pi^2+1)}$

Answer 1.9

a. $u(x) = \frac{1}{6}(\pi^3 - x^3) + \frac{1}{2}(x^2 - \pi^2)$

b. $R(x) = -U''(x) - x + 1 = \frac{1}{4}\xi_0 \cos\frac{x}{2} + \frac{9}{4}\xi_1 \cos\frac{3x}{2}$

c. $\xi_0 = 8(2\pi - 6)/\pi$ and $\xi_1 = \frac{8}{9}(\frac{2}{9} - \frac{2}{3}\pi)/\pi$

Answer 1.10 $U(x) = (16\sin x + \frac{16}{27}\sin 3x)/\pi^3 + 2x^2/\pi^2$

Answer 1.11 Check the conditions required for a Vector space.

Answer 1.12

$$\Pi_1 f(x) = f(a)\frac{2x-a-b}{a-b} + f(\frac{a+b}{2})\frac{2(x-a)}{b-a}.$$

Answer 1.13 Hint: Use Theorem 5.1 from PDE Lecture Notes.

Answer 1.14

$$\pi_4\left(e^{-8x^2}\right) \approx 0.25x^4 - 1.25x^2 + 1.$$

Answer 1.15 For example we may choose the following basis:

$$\varphi_{i,j}(x) = \begin{cases} 0, & x \in [x_{i-1}, x_i], \\ \lambda_{i,j}(x), & i = 1, \ldots, m+1, \quad j = 0, 1, 2. \end{cases}$$

$$\lambda_{i,0}(x) = \frac{(x-\xi_i)(x-x_i)}{(x_{i-1}-\xi_i)(x_{i-1}-x_i)}, \quad \lambda_{i,1}(x) = \frac{(x-x_{i-1})(x-x_i)}{(\xi_i - x_{i-1})(\xi_i - x_i)},$$

$$\lambda_{i,2}(x) = \frac{(x-x_{i-1})(x-\xi_i)}{(x_i - x_{i-1})(x_i - \xi_i)}, \qquad \xi_i \in (x_{i-1}, x_i).$$

Answer 1.16 Trivial.

Answer 1.17 Hint: Use Taylor expansion of f about $x = \frac{x_1+x_2}{2}$.