TMA682, Extra Excercises in Stability and Error estimates

1. Consider the initial value problem

$$\dot{u}(t) + au(t) = 0, \quad t > 0, \quad u(0) = u_0, \quad (a = \text{ constant}).$$

Assume a constant time step k and verify the iterative formulas for $dG(0)$ and $cG(1)$ approximations U and $\tilde{U}$, respectively: i.e.

$$U_n = \Big(\frac{1}{1+ak}\Big)^n u_0, \qquad \tilde{U}_n = \Big(\frac{1-ak/2}{1+ak/2}\Big)^n u_0.$$

2. Prove an a priori and an a posteriori error estimate for a finite element method (for example $cG(1)$) for the problem

$$-u'' + \alpha u = f, \quad \text{in } I = (0,1), \qquad u(0) = u(1) = 0,$$

where the coefficient $\alpha = \alpha(x)$ is a bounded positive function on I.

3. a) Formulate a $cG(1)$ method for the problem

$$\begin{cases} (a(x)u'(x))' = 0, & 0 < x < 1, \\ a(0)u'(0) = u_0, & u(1) = 0. \end{cases}$$

and give an a posteriori error estimate.

b) Compute the approximate solution in a) for a uniform partition of $I = [0,1]$ into 4 intervals and

$$a(x) = \begin{cases} 1/4, & x < 1/2, \\ 1/2, & x > 1/2. \end{cases}$$

c) Show that, with these special choices, the computed solution is equal to the exact one, i.e. the error is equal to 0.

4. Let $\|\cdot\|$ denote the $L_2(0,1)$-norm. Consider the problem

$$\begin{cases} -u'' = f, & 0 < x < 1, \\ u'(0) = v_0, & u(1) = 0. \end{cases}$$

a) Show that $|u(0)| \leq \|u'\|$ and $\|u\| \leq \|u'\|$.

b) Use a) to show that $\|u'\| \leq \|f\| + |v_0|$.

5. Let $\|\cdot\|$ denote the $L_2(0,1)$-norm. Consider the following heat equation

$$\begin{cases} \dot{u} - u'' = 0, & 0 < x < 1, & t > 0, \\ u(0,t) = u_x(1,t) = 0, & & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. & \end{cases}$$

a) Show that the norms: $||u(\cdot,t)||$ and $||u_x(\cdot,t)||$ are non-increasing in time. $||u|| = \left(\int_0^1 u(x)^2\,dx\right)^{1/2}$.

b) Show that $||u_x(\cdot,t)|| \to 0$, as $t \to \infty$.

c) Give a physical interpretation for a) and b).

6. Consider the problem

$$-\varepsilon u'' + xu' + u = f, \quad \text{in } I = (0,1), \qquad u(0) = u'(1) = 0,$$

where ε is a positive constant, and $f \in L_2(I)$. Prove that

$$||\varepsilon u''|| \leq ||f||.$$

7. Give an a priori error estimate for the following problem:

$$(au_{xx})_{xx} = f, \quad 0 < x < 1, \qquad u(0) = u'(0) = u(1) = u'(1) = 0,$$

where $a(x) > 0$ on the interval $I = (0,1)$.

8. Prove an a priori error estimate for the finite element method for the problem

$$-u''(x) + u'(x) = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

9. (a) Prove an *a priori* error estimate for the $cG(1)$ approximation of the boundary value problem

$$-u'' + cu' + u = f \quad \text{in } I = (0,1), \qquad u(0) = u(1) = 0,$$

where $c \geq 0$ is constant.

(b) For which value of c is the *a priori* error estimate optimal?

10. We modify problem 2 above according to

$$-\varepsilon u'' + c(x)u' + u = f(x) \quad 0 < x < 1, \qquad u(0) = u'(1) = 0,$$

where ε is a positive constant, the function c satisfies $c(x) \geq 0,\ c'(x) \leq 0$, and $f \in L_2(I)$. Prove that there are positive constants $C_1,\ C_2$ and C_3 such that

$$\sqrt{\varepsilon}||u'|| \leq C_1||f||, \quad ||cu'|| \leq C_2||f||, \qquad \text{and} \quad \varepsilon||u''|| \leq C_3||f||,$$

where $||\cdot||$ is the $L_2(I)$-norm.

11. Show that for a continuously differentiable function v defined on $(0,1)$ we have that

$$||v||^2 \leq v(0)^2 + v(1)^2 + ||v'||^2.$$

Hint: Use partial integration for $\int_0^{1/2} v(x)^2\,dx$ and $\int_{1/2}^1 v(x)^2\,dx$ and note that $(x - 1/2)$ has the derivative 1.

12. Determine the solution for the wave equation

$$\begin{cases} \ddot{u} - c^2 u'' = f, & x > 0, \qquad\qquad t > 0, \\ u(x,0) = u_0(x), & u_t(x,0) = v_0(x), \quad x > 0, \\ u_x(1,t) = 0, & \qquad\qquad\qquad t > 0, \end{cases}$$

in the following cases:

a) $f = 0$.

b) $f = 1, \quad u_0 = 0, \quad v_0 = 0.$