TMA683: INTRODUCTION TO CONVOLUTIONS

1. Convolutions

A *convolution* is a mathematical operation taking two functions as input, and producing a third function as output, much like addition or multiplication operations but with a more involved definition. As we will see below, convolutions have interesting applications in connection with Laplace transforms because of their simple transforms.

Definition 1.1. *The* convolution *of two functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined by*

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \tag{1}$$

if the integral is bounded.

In the context of Laplace transforms, we assume that $f(t) = 0$ and $g(t) = 0$ for $t < 0$. In this case, we may reduce the limits of integration to where the integrand is non-zero, and the convolution becomes

$$(f * g)(t) = \int_{0}^{t} f(t-\tau)g(\tau)d\tau \ \theta(t), \tag{2}$$

where $\theta(t)$ is the Heaviside step function, i.e. $(f * g)(t) = 0$ for $t < 0$.

The convolution satisfies some basic relations.

Theorem 1.1.

a) The convolution $f * g$ is a bi-linear operation, i.e. for all $\alpha, \beta \in \mathbb{R}$

$$(\alpha f + \beta g) * h = \alpha(f * h) + \beta(g * h)$$

and similar in the second argument.

The convolution also satisfies

b) $f * g = g * f$

c) $(f * g) * h = f * (g * h)$

Proof. The relations are direct consequences of the definition and a) and c) are left as exercises. As for b),

$$\begin{aligned}(f * g)(t) &= \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau = \begin{Bmatrix} \eta = t-\tau \\ d\eta = -d\tau \end{Bmatrix} = -\int_{\infty}^{-\infty} f(\eta)g(t-\eta)d\eta \\ &= \int_{-\infty}^{\infty} f(\eta)g(t-\eta)d\eta = (g * f)(t)\end{aligned}$$

□

One of the main reasons for using convolutions is their simple Laplace transform, namely

Theorem 1.2 (Convolution theorem). *If $f(t) = 0$ for $t < 0$ and $g(t) = 0$ for $t < 0$ (f and g are* causal*), with $\mathcal{L}[f] = F$ and $\mathcal{L}[g] = G$, and if there exist M and a so that $|f(t)| \le Me^{at}$ and $|g(t)| \le Me^{at}$, we have*

$$\mathcal{L}[f * g](s) = F(s)G(s). \tag{3}$$

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Proof.

$$
\begin{aligned}
\mathcal{L}[f*g](s) &= \int_0^\infty e^{-st}\left(\int_0^t f(t-\tau)g(\tau)d\tau\right)dt = \int_0^\infty \int_0^t e^{-st} f(t-\tau)g(\tau)d\tau dt \\
&= \left\{\begin{matrix}\text{switch order of integration,} \\ \text{see figure 1}\end{matrix}\right\} = \int_0^\infty g(\tau)\left(\int_\tau^\infty e^{-st} f(t-\tau)dt\right)d\tau \\
&= \int_0^\infty e^{-s\tau} g(\tau)\left(\int_\tau^\infty e^{-s(t-\tau)} f(t-\tau)dt\right)d\tau \\
&= \{\text{set } r = t-\tau \text{ in the inner integral}\} \\
&= \int_0^\infty e^{-s\tau} g(\tau)\left(\int_0^\infty e^{-sr} f(r)dr\right)d\tau = F(s)G(s)
\end{aligned}
$$

□

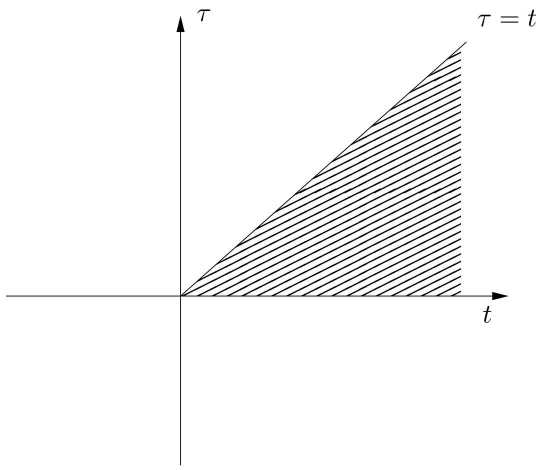

FIGURE 1. The integration area in the proof of theorem 1.2.

EXERCISES

1. Compute $(f*g)(t)$ when

   a) $f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$ and $g(t) = t\theta(t)$

   b) $f(t) = (e^{-t} - e^{-2t})\theta(t)$ and $g(t) = e^t\theta(t)$

2. Use the convolution theorem to compute the inverse Laplace transform of

   a) $F(s) = \dfrac{1}{(s^2+1)(s^2+4)}$ (Hint: $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$)

   b) $F(s) = \dfrac{1}{s^2(s^2+9)}$

Answers

1. a) $(f*g)(t) = \begin{cases} 0, & t < 0 \\ t^2/2, & 0 \leq t < 1 \\ t - 1/2, & t \geq 1 \end{cases}$

   b) $(f*g)(t) = \frac{1}{6}(e^t - 3e^{-t} + 2e^{-2t})\theta(t)$

2. a) $\frac{1}{3}\sin t - \frac{1}{6}\sin 2t$, b) $\frac{1}{9}t - \frac{1}{27}\sin 3t$