TMA682, Extra Excercise in Fourier series

The function f in the following exercises is assumed to be 2π-periodic, unless otherwise explicitly stated.

1. Find the Fourier series expansions of

$$\text{(a)} \quad f(x) = |\sin x|, \qquad \text{(b)} \quad f(x) = |\cos x|.$$

answer: (a) $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$, (b) $|\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \cos 2nx}{4n^2-1}$.

2. Use the Fourier series expansion for $f(x) = x^2$, $(-\pi < x < \pi)$:

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx,$$

to show that

(a)

$$x^3 - \pi^2 x = 12\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin nx, \qquad -\pi < x < \pi$$

(b)

$$x^4 - 4\pi^2 x^2 = 48\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \cos nx - \frac{7\pi^4}{15}, \qquad -\pi < x < \pi$$

(c)

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

We define the even and odd parts of a function $f(x)$ by

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)] \quad \text{and} \quad f_o(x) = \frac{1}{2}[f(x) - f(-x)].$$

3. Show that $f_e(x)$ is an even function, and $f_o(x)$ is an odd function.

4. What are the even and odd parts of the following function?

$$f(x) = \begin{cases} x^2, & x < 0 \\ e^{-x}, & x > 0. \end{cases}$$

answer:

$$f_e(x) = \frac{1}{2}\begin{cases} x^2 + e^{x}, & x < 0 \\ x^2 + e^{-x}, & x > 0. \end{cases} \qquad f_o(x) = \frac{1}{2}\begin{cases} x^2 - e^{x}, & x < 0 \\ e^{-x} - x^2, & x > 0. \end{cases}$$

5. The function $f(x) = 2x,\ 0 \le x \le 1$ is periodic with period $P = 1$.

   (a) Find the Fourier series expansion of $f(x)$.

   (b) Use the result in (a) to compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

answer: (a) $f(x) \sim 1 - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin 2n\pi x.$ (b) $\pi^2/6.$

6. Assume that the function $f(x) = x^2,\ 0 < x < 2$ is 2-periodic. Find the Fourier series expansion of $f(x)$.

   answer: $f(x) = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x.$

7. (a) Find the Fourier series expansion of the 2-periodic function f defined in $[-1, 1]$:

$$f(x) = \begin{cases} 1, & |x| \le 1/2 \\ 0, & 1/2 < |x| \le 1 \end{cases}$$

   (b) What is the series sum in the discontinuity points?

   answer: (a) $f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \cos(2n-1)\pi x.$ (b) $1/2.$

8. Assume that the function $f(x) = x,\ 0 < x < 2$ is 2-periodic.

   (a) Find the complex Fourier series expansion of $f(x)$.

   (b) Use (a) to give the real (cosinus-sinus form) Fourier series expansion of $f(x)$.

   (c) Find all solutions to the differential equation

$$y''(x) - y(x) = f(x).$$

   answer: (a) $f(x) = 1 - \sum_{n\neq 0} \frac{1}{in\pi} e^{in\pi}$. (b) $f(x) = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x$.
   (c) $y(x) = y_h(x) + y_p(x), \quad y_h(x) = Ae^x + Be^{-x}$,
   $y_p(x) = \sum_{n=-\infty}^{\infty} y_n e^{in\pi x}, \quad y_0 = -1, \quad (1 + n^2\pi^2) y_n = \frac{1}{in\pi}, \quad n \neq 0.$

9. The function $f(x) = |x|^3,\ |x| \leq 2$ is 4-periodic. Find the Fourier series expasion for both f and f'.

   answer:

$$f(x) = 2 + \frac{48}{\pi^4} \sum_{n=1}^{\infty} \frac{2 + (-1)^n (n^2\pi^2 - 2)}{n^4} \cos \frac{n\pi}{2} x.$$

$$f'(x) = -\frac{24}{\pi^3} \sum_{n=1}^{\infty} \frac{2 + (-1)^n (n^2\pi^2 - 2)}{n^3} \sin \frac{n\pi}{2} x.$$

10. The data function $f(x) = x(2 - x),\ 0 \leq x < 2$ is 2-periodic. Find a 2-peridic solution for the differential equation

$$y''(x) + y'(x) + 2y(x) = f(x),$$

   as a complex Fourier serirs.

   answer:

$$y(x) = \frac{1}{3} + 2 \sum_{n\neq 0} \frac{e^{in\pi x}}{n^2\pi^2(n^2\pi^2 - in\pi - 2)}.$$