TMA682, Extra Excercise in Separation of Variables

1. Solve the boundary value problem (Laplace's equation)

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 2, \qquad 0 < y < \infty, \\ u(0, y) = u_x(2, y) = 0, & \lim_{y\to\infty} u(x, y) = 0, \\ u(x, 0) = 0, \quad 0 < x < 1, & u(x, 0) = 1, \quad 1 < x < 2, \end{cases}$$

answer:

$$u(x, y) = \sum_{n=0}^{\infty} \frac{\cos \alpha_n}{\alpha_n} e^{-\alpha_n y} \sin \alpha_n x, \qquad \alpha_n = (n + \frac{1}{2})\frac{\pi}{2}.$$

2. Solve the boundary value problem (Laplace's equation)

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, \qquad 0 < y < b, \\ u(0, y) = u_x(a, y) = 0, & \\ u(x, 0) = 0, & u(x, b) = x^2 - 2ax. \end{cases}$$

answer:

$$u(x, y) = \frac{-4a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^3} \sin(n + \frac{1}{2})\frac{\pi x}{a} \cdot \frac{\sinh(n + \frac{1}{2})\frac{\pi y}{a}}{\sinh(n + \frac{1}{2})\frac{\pi b}{a}}.$$

3. Solve the inhomogeneous boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = y, & x > 0, \qquad 0 < y < 1 \\ u(x, 0) = u(x, 1) = 0, & \\ u(0, y) = y - y^3, & u \text{ is bounded as } x \to \infty. \end{cases}$$

answer:

$$u(x, y) = \frac{1}{6}(y^3 - y) + \sum_{n=1}^{\infty} \frac{7(-1)^n}{2(n\pi)^3} e^{-n\pi x} \sin n\pi y.$$

4. Solve the initial-boundary value problem (heat equation)

$$\begin{cases} u_t = u_{xx}, & 0 < x < \pi, \qquad t > 0 \\ u(0,t) = 1, & u(\pi,t) = -1, \\ u(x,0) = \cos x. \end{cases}$$

answer:

$$u(x,t) = 1 - \frac{2x}{\pi} + \sum_{n=1}^{\infty} \Big((-1)^{k+1} - 1 \Big) \frac{2}{k(k^2-1)\pi} e^{-k^2 t} \sin kx.$$

5. Solve the following initial-boundary value problem (wave equation)

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < \ell, & t > 0, \quad c > 0 \\ u_x(0,t) = 0, & u_x(\ell,t) = 0, \\ u(x,0) = 1, & u_t(x,0) = -\cos \frac{\pi}{\ell} x. \end{cases}$$

answer:

$$u(x,t) = 1 - \frac{\ell}{\pi c} \cos \frac{\pi x}{\ell} \sin \frac{\pi c t}{\ell}.$$

6. Solve the inhomogeneous initial-boundary value problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < \ell, \qquad t > 0, \quad c > 0 \\ u(0,t) = 0, & u(\ell,t) = 1, \\ u(x,0) = 2\frac{x}{\ell} - 1. \end{cases}$$

answer:

$$u(x,t) = \frac{x}{\ell} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{n^2 \pi^2}{\ell^2} t} \sin \frac{n\pi}{\ell} x.$$

7. Solve the inhomogeneous problem

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < 1, \qquad t > 0, \\ u(0,t) = 1, & u(1,t) = 0, \\ u(x,0) = 0, & u_t(x,0) = 0. \end{cases}$$

answer:

$$u(x,t) = 1 - x - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi t \sin n\pi x.$$

8. Solve the initial-boundary value problem

$$\begin{cases} u_t = u_{xx} - \sin\frac{2\pi x}{\ell}, & 0 < x < \ell, \qquad t > 0 \\ u(x,0) = u(0,t) = u(\ell,t) = 0. \end{cases}$$

answer:

$$u(x,t) = \left(\frac{\ell}{2\pi}\right)^2 \left(e^{-\frac{4\pi^2 t}{\ell^2}} - 1\right) \sin\frac{2\pi x}{\ell}.$$

9. Let $u(x,t)$ be the solution for the following problem

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < \pi, \qquad t > 0, \quad c > 0 \\ u(0,t) = u(\pi,t) = 0, & \\ u(x,0) = 0, & u_t(x,0) = g(x). \end{cases}$$

Show that for $t > 0$,

$$\int_0^\pi |u_t(x,t)|^2 \leq \int_0^\pi |g(x)|^2.$$

10. Solve the differential equation

$$\begin{cases} u_{tt} = u_{xx} - \pi^2 u, & 0 < x < 1, \qquad t > 0, \quad c > 0 \\ u(0,t) = u(1,t) = 0, & \\ u_t(x,0) = 0, & u(x,0) = \cos(\pi x), \quad 0 < x < 1 \end{cases}$$

answer:

$$u(x,t) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{k}{4k^2 - 1} \sin(2k\pi x) \cos\left(\sqrt{4k^2 + 1}\pi t\right).$$

11. A substance is diffusing in a straight cylindrical pipe of length ℓ with closed intersections. Suppose that the symmetry axis of the cylinder is aligned with the x-axis. If the density of substance at the point x at time t is denoted by $\rho(x,t)$, then $\rho(x,t)$ satisfies the diffusion equation

$$\rho_t = C\rho_{xx},$$

where C is a constant. Determine $\rho(x,t)$ if $\rho(x,0)$ varies linearly from 0 to ρ_0 as x goes from 0 to ℓ.

answer:

$$\rho(x,t) = \frac{\rho_0}{2} - \frac{4\rho_0}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-\frac{C(2n-1)^2\pi^2 t}{\ell^2}} \cos\frac{(2n-1)\pi x}{\ell}.$$

12. Solve the following inhomogeneous initial-boundary value problem

$$\begin{cases} u_t = u_{xx} + e^{-t} \sin 3x, & 0 < x < \pi, \quad t > 0 \\ u(0,t) = 0, \quad u(\pi,t) = 1, & u(x,0) = 2. \end{cases}$$

answer:

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{2-(-1)^n}{n} e^{-n^2 t} \sin nx + \frac{x}{\pi} + \frac{1}{8}\left(e^{-t} - e^{-9t}\right) \sin 3x.$$

13. Compute the stationary temperature $u(x,y)$ in the square plate

$$A = \{(x,y) : 0 \le x \le 100, \ \ 0 \le y \le 100\},$$

if the side $y = 100$ is kept at temperature $100^\circ\, C$ and all other sides at the temperature $0^\circ\, C$. Determine, in particular, the stationary temperature at the midpoint of the plate.

Hint: The stationary heat equation satisfies Laplace's equation

answer:

$$u(x,y) = \frac{400}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)\sinh(2k-1)\pi} \sinh \frac{(2k-1)\pi y}{100} \sin \frac{(2k-1)\pi x}{100},$$

$$u(50,50) = \frac{200}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)\cosh \frac{(2k-1)\pi}{2}} \approx 25^\circ\, C.$$

14. a) Determine the function $u(x,t)$ satisfying:

$$\begin{cases} 4u_{xx} = u_{tt}, & 0 < x < 2, & t > 0 \\ u(x,0) = (1-x)\theta(1-x) & u_t(x,0) = 0, & 0 < x < 2 \\ u(0,t) = u(2,t) = 0, & & t > 0. \end{cases}$$

b) Determine $u(x, \frac{1}{2})$.

answer:

$$u(x,t) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right) \cos(n\pi t) \sin\frac{n\pi x}{2}.$$

$$u(x, \frac{1}{2}) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^k}{k} \sin k\pi x.$$

15. Solve the problem

$$\begin{cases} u_{xx} + u_{yy} = 1, & 0 < x < 1, & 0 < y < 1 \\ u(x,0) = 0, & u_y(x,0) = 0, & 0 < x < 1 \\ u(0,y) = 0, & u(1,y) = y^2 - 2y. & \end{cases}$$

answer:

$$u(x,y) = \frac{1}{2}(y^2 - y) + \frac{2}{\pi^3} \sum_{n=0}^{\infty} \frac{\sinh(n+\frac{1}{2})\pi(1-x) - \sinh(n+\frac{1}{2})\pi x}{(n+\frac{1}{2})^3 \sinh(n+\frac{1}{2})\pi} \sin(n+\frac{1}{2})\pi y.$$