

## Chap 3: PDEs

Superposition principle:

Consider the homogeneous ODE,  $U''(x) = 0$ .

Since  $U_1(x) = 2x$  and  $U_2(x) = -x + 3$  are solutions to the ODE, so  $\forall c_1, c_2 \in \mathbb{R}$ , function  $U(x) = c_1 U_1(x) + c_2 U_2(x)$  is also a solution. Indeed,

$$U''(x) = c_1 U_1''(x) + c_2 U_2''(x) = c_1(0) + c_2(0) = 0$$

Consider the homogeneous PDE (Laplace eq.):

$$U_{xx}(x, y) + U_{yy}(x, y) = 0$$

Since  $U_1(x, y) = x^2 - y^2$ ,  $U_2(x, y) = e^x \cos y$  both satisfy the Laplace eq., so  $\forall c_1, c_2 \in \mathbb{R}$ , function

$$U(x, y) = c_1 U_1(x, y) + c_2 U_2(x, y)$$

is also a solution. Indeed:

$$\{ \begin{aligned} U_{xx} &= c_1(2) + c_2(e^x \cos y) \\ U_{yy} &= c_1(-2) + c_2(-e^x \cos y) \end{aligned}$$

$$\Rightarrow U_{xx} + U_{yy} = 0$$

Recall: exact solution of some linear ODE,

$$y'(t) + ky(t) = 0 \implies y(t) = Ce^{-kt}$$

$$y''(t) + \mu^2 y(t) = 0 \stackrel{\mu > 0}{\implies} y(t) = A \cos \mu t + B \sin \mu t$$

$$y''(t) - \mu^2 y(t) = 0 \stackrel{\mu > 0}{\implies} y(t) = Ae^{\mu t} + Be^{-\mu t}$$

## 3.2 Separation of Variables

It is a method to find the exact solution of some linear PDE.

Based on considering

$$u(x,t) = X(x)T(t)$$

### 3.2.2 The Heat Eq.

We solve the IVP (homogeneous heat eq.).  $u=u(x,t)$

$k$  a positive constant

$$\text{DE } \left\{ \begin{array}{l} \dot{u} - ku'' = 0, \\ 0 < x < L, t > 0 \end{array} \right.$$

$$\text{BC } \left\{ \begin{array}{l} u(0,t) = u(L,t) = 0 \\ t > 0 \end{array} \right.$$

$$\text{IC } \left\{ \begin{array}{l} u(x,0) = f(x) \\ 0 < x < L \end{array} \right.$$

Note that if  $f(x)=0$ , then the only solution is  $u(x,t)=0$ . If  $f(x)\neq 0$  then there is only one solution that tends to zero in time.

$$\lim_{t \rightarrow \infty} u(x,t) = 0$$

Substitute in DE:

$$X(x) \dot{T}(t) - k X''(x) T(t) = 0$$

$$\Rightarrow X(x) \dot{\frac{T(t)}{T(t)}} = k X''(x) \frac{T(t)}{T(t)}$$

$$\xrightarrow[\text{Divide by } k X(x) T(t)]{} \frac{\dot{T}(t)}{k T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

$$\Rightarrow \left\{ \begin{array}{l} X''(x) = \lambda X(x) \\ \dot{T}(t) = k \lambda T(t) \end{array} \right. \xrightarrow{k \lambda t} T(t) = e^{k \lambda t}$$

Three cases for  $\lambda$ :

$$\textcircled{1} \quad \lambda > 0 \Rightarrow T(t) \xrightarrow[t \rightarrow \infty]{} \infty \quad \text{when } t \rightarrow \infty \quad (\lim_{t \rightarrow \infty} T(t) = \infty)$$

$\Rightarrow u(x,t) = X(x)T(t)$  increases with  $t$   
but there is no source of heat

②  $\lambda = 0 \Rightarrow T(t) = C^0 = C$  but the temperature should tends to 0.

③  $\lambda = -\mu^2 < 0$ , then  $u(x,t) \xrightarrow{t \rightarrow \infty} 0$ , so this is correct

Now we solve the first ODE, that is

$$X''(x) = -\mu^2 X(x)$$

$$\Rightarrow X(x) = A \cos \mu x + B \sin \mu x$$

to find A and B we should use the B.C.

$$0 = u(0, t) = X(0) T(t) \Rightarrow \begin{cases} T(t) = 0 \Rightarrow T = 0 \Rightarrow u = X T = 0 \text{ (No)} \\ \text{or} \\ X(0) = 0 \end{cases}$$

$$0 = u(L, t) = X(L) T(t) \Rightarrow \begin{cases} T(t) = 0 \Rightarrow u = 0 \text{ (No)} \\ \text{or} \\ X(L) = 0 \end{cases}$$

Now using this BC.

$$X(0) = 0 \Rightarrow A \cos 0 + B \sin 0 = 0 \Rightarrow A = 0 \Rightarrow X(t) = B \sin \mu x$$

$$X(L) = 0 \Rightarrow B \sin \mu L = 0 \Rightarrow \begin{cases} B = 0 \Rightarrow X = 0 \Rightarrow u = X T = 0 \text{ (No)} \\ \sin \mu L = 0 \Rightarrow \mu = \frac{n\pi}{L}, n=1, 2, \dots \end{cases}$$

$$\text{Thus } X_n(x) = B_n \sin \frac{n\pi}{L} x, n=1, 2, \dots$$

and

$$T_n(x) = C_n e^{-k \frac{n^2 \pi^2}{L^2} t} = C_n e^{-k \frac{n^2 \pi^2}{L^2} t}$$

So for any  $n=1, 2, \dots$

$$u_n(x, t) = X_n(x) T_n(t) = C_n e^{-k \frac{n^2 \pi^2}{L^2} t} \sin \frac{n\pi}{L} x$$

is a solution of DE that satisfies the B.C.

By superposition principle the solution is

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} C_n e^{-k \frac{n^2 \pi^2}{L^2} t} \sin \frac{n\pi}{L} x$$

To find the parameters  $C_n$ , we use the I.C.

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{L} x = f(x)$$

So  $C_n$  are the coefficients of Fourier sine series of  $f$ . That is

$$C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx, \quad n=1, 2, \dots$$

Ex. Use the separation of variables to solve ( $u=u(x,t)$ )

$$\begin{cases} \ddot{u} - u'' = 0 & 0 < x < 1, t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,t) = x & 0 < x < 1 \end{cases}$$

$$\text{Look for } u(x,t) = X(x)T(t)$$

$$\stackrel{\text{DE}}{\Rightarrow} X\dot{T} - X''T = 0$$

$$\Rightarrow X\dot{T} = X''T$$

$$\text{Divide by } X\dot{T} \Rightarrow \frac{\dot{T}}{T} = \frac{X''}{X} = \lambda \quad \begin{cases} \lambda = -\mu^2 \\ X'' = -\mu^2 X \end{cases} \quad \begin{cases} \dot{T} = -\mu^2 T \\ \otimes \end{cases}$$

First solve  $X'' = -\mu^2 X$

$$\begin{cases} X(0) = 0, X(1) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X(x) = A \cos \mu x + B \sin \mu x \\ X(0) = 0, X(1) = 0 \end{cases}$$

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B \sin \mu x$$

$$X(1) = 0 \Rightarrow B \sin \mu = 0 \Rightarrow \mu = n\pi, \quad n=1, 2, \dots$$

Hence for any  $n=1, 2, \dots$  choose  $\lambda = n^2 \pi^2$

$$X_n(x) = B_n \sin n\pi x$$

From the second ODE in  $\otimes$  with  $\mu = n\pi$ , we have

$$T_n(t) = C_n e^{-n^2 \pi^2 t}, \quad n=1, 2, \dots$$

Thus for any  $n=1, 2, \dots$

$$u_n(x,t) = X_n(x)T_n(t) = C_n e^{-n^2 \pi^2 t} \sin n\pi x$$

is a solution, and by superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{n^2 \pi^2 t}{L}} \sin n\pi x$$

Using the IC,

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin n\pi x = x$$

$$\Rightarrow C_n = \frac{2}{L} \int_0^L f(x) \sin n\pi x dx = 2 \int_0^1 x \sin n\pi x dx$$

$$= \frac{2(-1)^{n+1}}{n\pi}, \quad n=1, 2, \dots$$

$$\text{So } u(x,t) = \underbrace{\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{n^2 \pi^2 t}{L}} \sin n\pi x}_{\text{Final Answer}}$$

Ex 17 Use the separation of variables to solve

$$\begin{cases} \ddot{u} - u' = 0 & 0 < x < 1, t > 0 \\ u'(0, t) = u'(1, t) = 0 & t > 0 \\ u(x, 0) = f(x) & 0 < x < 1 \end{cases}$$

where

$$f(x) = \begin{cases} 0 & 0 < x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x < 1 \end{cases}$$

$$\text{Put } u(x, t) = X(x)T(t)$$

$$\Rightarrow X\ddot{T} - X''T = 0 \Rightarrow X\ddot{T} = X''T \Rightarrow \frac{\ddot{T}}{T} = \frac{X''}{X} = \lambda \stackrel{\lambda = -\mu^2}{\Rightarrow} \begin{cases} X'' = -\mu^2 X \\ \frac{\ddot{T}}{T} = -\mu^2 \end{cases} \quad \textcircled{*}$$

$$\text{First solve } \begin{cases} X'' = -\mu^2 X \\ X'(0) = 0, X'(1) = 0 \end{cases} \Rightarrow \begin{cases} X(x) = A \cos \mu x + B \sin \mu x \\ X'(0) = 0, X'(1) = 0 \end{cases}$$

$$X'(0) = 0 \Rightarrow \underline{X(x) = -AM \sin \mu x + BM \cos \mu x} \quad BM = 0 \Rightarrow B = 0 \Rightarrow X(x) = -AM \sin \mu x$$

$$X'(1) = 0 \Rightarrow -AM \sin \mu = 0 \stackrel{\mu \neq 0}{\Rightarrow} A = 0 \Rightarrow \mu = n\pi, n = 1, 2, \dots$$

$$\text{So } X_n(x) = A_n \cos n\pi x, n = 1, \dots$$

and from the second ODE in  $\textcircled{*}$

$$T_n(t) = C_n e^{-n^2 \pi^2 t}, n = 1, 2, \dots$$

Hence

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-n^2 \pi^2 t} \cos n\pi x$$

Now using the IC.

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \cos n\pi x = f(x)$$

$$\begin{aligned} \Rightarrow C_n &= \frac{2}{L} \int_0^L f(x) \cos n\pi x dx = 2 \int_0^{\frac{1}{2}} (0) \cos n\pi x dx + 2 \int_{\frac{1}{2}}^1 1 \cos n\pi x dx \\ &= \frac{2}{n\pi} \left( \underbrace{\sin n\pi}_{=0} - \sin \frac{n\pi}{2} \right) = -\frac{2}{n\pi} \sin \frac{n\pi}{2} \end{aligned}$$

Hence

$$u(x, t) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2} e^{-n^2 \pi^2 t} \cos n\pi x.$$

### 3.2.3 The Wave Eq.

We solve the IVP (homogeneous wave eq.)  $u=u(x,t)$

$\nabla C > 0$  a constant

$$DE \quad \ddot{u} - c^2 \ddot{u} = 0 \quad 0 < x < L, t > 0$$

$$BC \quad u(0,t) = u(L,t) = 0 \quad t > 0$$

$$IC \quad u(x,0) = f(x) \quad 0 < x < L$$

$$\dot{u}(x,0) = g(x) \quad 0 < x < L$$

$$\text{Look for } u(x,t) = X(x)T(t)$$

$$DE \Rightarrow X \ddot{T} - c^2 X \ddot{T} = 0 \Rightarrow X \ddot{T} = c^2 X \ddot{T} \xrightarrow{\text{Divide}} \frac{\ddot{T}}{c^2 T} = \frac{\ddot{X}}{X} = 1$$

$$\lambda = -\mu^2 \quad \ddot{X}(\lambda) = -\mu^2 X(\lambda)$$

$$\ddot{T}(\lambda) = -c^2 \mu^2 T(\lambda)$$

$$\text{First Solve } \begin{cases} \ddot{X} = -\mu^2 X \\ X(0) = X(L) = 0 \end{cases} \Rightarrow \begin{cases} X(x) = A \cos \mu x + B \sin \mu x \\ X(0) = X(L) = 0 \end{cases}$$

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B \sin \mu x$$

$$X(L) = 0 \Rightarrow B \sin \mu L = 0 \xrightarrow[B \neq 0]{} \sin \mu L = 0 \Rightarrow \mu L = n\pi, n=1, 2, \dots \Rightarrow \mu = \frac{n\pi}{L}, n=1, 2, \dots$$

So choose  $\lambda = \frac{n\pi}{L}$

$$X_n(x) = B_n \sin \frac{n\pi}{L} x, n=1, 2, \dots$$

$$\text{and } T_n(t) = C_n \cos \frac{n\pi}{L} t + D_n \sin \frac{n\pi}{L} t, n=1, 2, \dots$$

Hence

$$u(x,t) = \sum_{n=1}^{\infty} \left( C_n \cos \frac{n\pi}{L} t + D_n \sin \frac{n\pi}{L} t \right) \sin \frac{n\pi}{L} x$$

$$\dot{u}(x,t) = \sum_{n=1}^{\infty} \left( -\frac{n\pi}{L} C_n \sin \frac{n\pi}{L} t + \frac{n\pi}{L} D_n \cos \frac{n\pi}{L} t \right) \sin \frac{n\pi}{L} x$$

Use the IC.

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{L} x = f(x)$$

$$\dot{u}(x,0) = \sum_{n=1}^{\infty} \frac{n\pi}{L} D_n \sin \frac{n\pi}{L} x = g(x)$$

So

$$\left\{ \begin{array}{l} C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx \\ \frac{C_n n\pi}{L} D_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x dx \end{array} \right. \Rightarrow \left\{ \begin{array}{l} C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx \\ D_n = \frac{2}{C_n n\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x dx \end{array} \right. \quad n=1, 2, \dots$$

Ex Use the separation of variables to solve

$$\left\{ \begin{array}{l} \ddot{U} - U'' = 0 \quad 0 < x < 1, \quad t > 0 \\ U(0, t) = U(1, t) = 0 \quad t > 0 \\ U(x, 0) = \sin \pi x, \quad \dot{U}(x, 0) = 0 \quad 0 < x < 1 \end{array} \right.$$

Look for  $U(x, t) = X(x)T(t)$

$$\text{DE} \Rightarrow X'' T - X'' T = 0 \Rightarrow X'' T = X'' T \xrightarrow{\text{Divide}} \frac{T}{X} = \frac{X''}{T} = \lambda$$
$$\lambda = -\mu^2 \xrightarrow{\sum} \left\{ \begin{array}{l} X'' = -\mu^2 X \\ T = -\mu^2 T \end{array} \right.$$

First  $\left\{ \begin{array}{l} X'' = -\mu^2 X \\ X(0) = X(1) = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} X(x) = A \cos \mu x + B \sin \mu x \\ X(0) = X(1) = 0 \end{array} \right.$

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B \sin \mu x$$

$$X(1) = 0 \Rightarrow B \sin \mu = 0 \xrightarrow[B \neq 0]{} \sin \mu = 0 \Rightarrow \mu = n\pi, \quad n=1, 2, \dots$$

So  $X_n(x) = \overbrace{B_n}^{=1} \sin n\pi x$

and

$$T_n(t) = C_n \cos n\pi t + D_n \sin n\pi t$$

Hence

$$\text{and } \left\{ \begin{array}{l} U(x, t) = \sum_{n=1}^{\infty} (C_n \cos n\pi t + D_n \sin n\pi t) \sin n\pi x \end{array} \right.$$

$$\left\{ \begin{array}{l} U(x, t) = \sum_{n=1}^{\infty} (-n\pi C_n \sin n\pi t + n\pi D_n \cos n\pi t) \sin n\pi x \end{array} \right.$$

Use I.C

$$\left\{ \begin{array}{l} u(x, 0) = \sum_{n=1}^{\infty} c_n \sin n\pi x = \sin \pi x \\ u_t(x, 0) = \sum_{n=1}^{\infty} n\pi D_n \sin n\pi x = 0 \end{array} \right.$$

$$\Rightarrow c_n = \frac{2}{1} \int_0^1 \sin \pi x \sin n\pi x dx \quad \left\{ \begin{array}{ll} n=1 & 1 \\ n \neq 1 & 0 \end{array} \right.$$
$$n\pi D_n = 0 \Rightarrow D_n = 0$$

Hence

$$u(x, t) = \sum_{n=1}^{\infty} c_n \cos n\pi t \sin n\pi x = \underline{\cos \pi t \sin \pi x}$$

1.  $\text{C}_2\text{H}_5\text{OH} + \text{O}_2 \rightarrow \text{CH}_3\text{CHO} + \text{H}_2\text{O}$

2.  $\text{CH}_3\text{CHO} + \text{O}_2 \rightarrow \text{CH}_3\text{COOH}$

3.  $\text{CH}_3\text{COOH} + \text{NaOH} \rightarrow \text{CH}_3\text{COONa} + \text{H}_2\text{O}$

4.  $\text{CH}_3\text{COONa} + \text{HCl} \rightarrow \text{CH}_3\text{COOH} + \text{NaCl}$

5.  $\text{CH}_3\text{COOH} + \text{C}_2\text{H}_5\text{OH} \rightarrow \text{CH}_3\text{COOC}_2\text{H}_5 + \text{H}_2\text{O}$

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