

### 3.2.4 Inhomogeneous Equations

We consider problems, that either the PDE or the B.C. are inhomogeneous

Solve the inhomogeneous heat equation:

$$\text{DE} \quad \begin{cases} \dot{u} - ku'' = q & 0 < x < L, t > 0 \\ \text{B.C.} \quad u(0,t) = \alpha, u(L,t) = \beta & t > 0 \end{cases}$$

$$\text{I.C.} \quad u(x,0) = f(x) \quad 0 < x < L$$

Where  $k, \alpha, \beta, q$  are constant. That is

$$\begin{cases} \dot{u}(x,t) - ku''(x,t) = q & 0 < x < L, t > 0 \\ u(0,t) = \alpha, u(L,t) = \beta & t > 0 \\ u(x,0) = f(x) & 0 < x < L \end{cases}$$

To solve the PDE, we split the solution (temperature)  $u(x,t)$  as the sum of a (transient) solution  $v(x,t)$  and a (steady state) solution  $s(x)$

$$u(x,t) = v(x,t) + s(x)$$

We impose the inhomogeneous terms to  $s(x)$ .

Put  $u(x,t) = v(x,t) + s(x)$  in the PDE, we have

$$\begin{cases} \dot{v}(x,t) + \dot{s}(x) - k(v''(x,t) + s''(x)) = q \\ v(0,t) + s(0) = \alpha, v(L,t) + s(L) = \beta \\ v(x,0) + s(x) = f(x) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{v}(x,t) - k\dot{v}(x,t) - ks''(x) = q \\ v(0,t) + s(0) = \alpha, v(L,t) + s(L) = \beta \\ v(x,0) + s(x) = f(x) \end{cases}$$

So we should solve an ODE and PDE:

$$\begin{array}{l} \text{(ODE)} \quad \begin{cases} s''(x) = -\frac{q}{k} \\ s(0) = \alpha, s(L) = \beta \end{cases} \end{array} \quad , \quad \begin{array}{l} \text{(PDE)} \quad \begin{cases} \dot{v} - kv'' = 0 \\ v(0,t) = v(L,t) = 0 \\ v(x,0) = f(x) - s(x) \end{cases} \end{array}$$

Solution of the ODE:

$$\begin{cases} s(x) = -\frac{q}{k} \frac{x^2}{2} + C_1 x + C_2 \\ s(0) = \alpha, s(L) = \beta \end{cases} \quad \Rightarrow \quad \begin{cases} s(0) = \alpha \Rightarrow C_2 = \alpha \\ s(L) = \beta \Rightarrow C_1 = \frac{1}{L} \left( \beta - \alpha + \frac{qL^2}{2k} \right) \end{cases}$$

$$\Rightarrow s(x) = \frac{-9x^2}{2k} + \frac{1}{L} \left( \beta - \alpha + \frac{9L^2}{2k} \right) x + \alpha$$

Then we solve (by the separation of variables) the PDE, (with initial value  $v(x,0) = f(x) - s(x)$ )

Ex Solve the PDE by the separation of variables method

$$\begin{cases} \dot{U} - U'' = 2 & 0 < x < 1, t > 0 \\ U(0,t) = 0, U(1,t) = 1 & t > 0 \\ U(x,0) = 1 & 0 < x < 1 \end{cases}$$

Split the solution as

$$\begin{aligned} U(x,t) &= V(x,t) + S(x) \\ \text{DE} \Rightarrow & \left( \dot{V}(x,t) + \overset{=0}{\cancel{S(x)}} \right) - \left( V''(x,t) + \overset{=0}{\cancel{S''(x)}} \right) = 2 \\ & \left\{ \begin{array}{l} V(0,t) + S(0) = 0, \quad V(1,t) + S(1) = 1 \\ V(x,0) + S(x) = 1 \end{array} \right. \\ \Rightarrow & \left\{ \begin{array}{l} \overset{=0}{\cancel{V(x,t) - V(0,t)}} - \overset{=0}{\cancel{S(x)}} = 2 \\ \overset{=0}{\cancel{V(0,t) + S(0)}} + \overset{=0}{\cancel{V(1,t) + S(1)}} = 1 \\ V(x,0) = 1 - S(x) \end{array} \right. \end{aligned}$$

So we solve

$$\begin{array}{l} (\text{ODE}) \begin{cases} -S''(x) = 2 & 0 < x < 1 \\ S(0) = 0, S(1) = 1 \end{cases}, \quad (\text{PDE}) \begin{cases} \dot{V} - V'' = 0 & 0 < x < 1, t > 0 \\ V(0,t) = V(1,t) = 0 & t > 0 \\ V(x,0) = 1 - S(x) \end{cases} \end{array}$$

$$\begin{aligned} \text{ODE: } & \begin{cases} S(x) = -x^2 + C_1 x + C_2 \\ S(0) = 0, S(1) = 1 \end{cases} \Rightarrow \begin{cases} S(0) = 0 \\ S(1) = 1 \end{cases} \begin{cases} C_2 = 0 \\ -1 + C_1 + C_2 = 1 \end{cases} \rightarrow C_1 = 2 \\ \Rightarrow & S(x) = -x^2 + 2x \end{aligned}$$

$$\text{PDE: look for } V(x,t) = X(x)T(t)$$

$$X\ddot{T} - X''T = 0 \Rightarrow \frac{\ddot{T}}{T} = \frac{X''}{X} = \lambda \Rightarrow \begin{cases} \lambda = -\mu^2 \\ \frac{\ddot{T}}{T} = -\mu^2 \end{cases} \begin{cases} X = -\mu^2 X \\ \ddot{T} = -\mu^2 T \end{cases}$$

$$\begin{array}{l} \text{First ODE: } \begin{cases} X'' = -\mu^2 X \\ X(0) = X(1) = 0 \end{cases} \Rightarrow \begin{cases} X(x) = A \cos \mu x + B \sin \mu x \\ X(0) = X(1) = 0 \end{cases} \\ X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B \sin \mu x \end{array}$$

$$X(1)=0 \Rightarrow B \sin \mu = 0 \xrightarrow{B \neq 0} \sin \mu = 0 \Rightarrow \mu = n\pi, n=1, 2, \dots$$

So

$$X_n(x) = B_n \sin n\pi x, n=1, 2, \dots$$

and

$$T_n(t) = C_n e^{-\mu t} = C_n e^{-n^2 \pi^2 t}, n=1, 2, \dots$$

$$\text{Hence } v(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 \pi^2 t} \sin n\pi x$$

Using the initial value

$$v(x,0) = 1 - S(x) = 1 + x^2 - 2x \Rightarrow \sum_{n=1}^{\infty} C_n \sin n\pi x = 1 + x^2 - 2x$$

$$\Rightarrow C_n = \frac{2}{1} \int_0^1 (1 + x^2 - 2x) \sin n\pi x dx, n=1, 2, \dots$$

$$= \left\{ \begin{array}{l} u = 1 + x^2 - 2x \\ du = (2x-2)dx \end{array} \right. \quad \left. \begin{array}{l} dv = \sin n\pi x dx \\ v = -\frac{1}{n\pi} \cos n\pi x \end{array} \right\}$$

$$= 2 \left[ (1 + x^2 - 2x) \left( -\frac{1}{n\pi} \cos n\pi x \right) \right]_0^1 + \frac{4}{n\pi} \int_0^1 \frac{(x-1) \cos n\pi x dx}{u} dv$$

$$= 2 \left( 0 - \left( -\frac{1}{n\pi} \right) \right) + \frac{4}{n\pi} \left\{ \left. \left( (x-1) \frac{1}{n\pi} \sin n\pi x \right) \right|_0^1 - \frac{1}{n\pi} \int_0^1 \sin n\pi x dx \right\}$$

$$= \frac{2}{n\pi} + \frac{4}{n\pi} \left( -\frac{1}{n\pi} \right) \left( -\frac{1}{n\pi} \right) \left( \cos n\pi x \right) \Big|_{x=0}^1$$

$$= \frac{2}{n\pi} + \frac{4}{n^3\pi^3} (-1)^n - 1$$

Thus

$$u(x,t) = v(x,t) + S(x)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left( \frac{2}{n\pi} + \frac{4}{n^3\pi^3} ((-1)^n - 1) \right) e^{-n^2 \pi^2 t} \sin n\pi x - x^2 + 2x$$

Ex Solve the following heat equation by the separation of variables.

$$\begin{cases} \dot{U} - U'' = 0 & 0 < x < \pi, t > 0 \\ U(0, t) = 1, U(\pi, t) = 0 & t > 0 \\ U(x, 0) = -\frac{1}{\pi}x & 0 < x < \pi \end{cases}$$

Split the solution as

$$U(x, t) = V(x, t) + S(x)$$

$$\Rightarrow \begin{cases} \dot{V}(x, t) - V''(x, t) = 0 \\ V(0, t) + S(0) = 1, V(\pi, t) + S(\pi) = 0 \\ V(x, 0) + S(x) = -\frac{1}{\pi}x \end{cases}$$

So, ~~we~~ solve

$$(ODE) \begin{cases} -S''(x) = 0 & 0 < x < \pi \\ S(0) = 1, S(\pi) = 0 \end{cases}$$

$$(PDE) \begin{cases} \dot{V} - V'' = 0 & 0 < x < \pi, t > 0 \\ V(0, t) = V(\pi, t) = 0 & t > 0 \\ V(x, 0) = -\frac{1}{\pi}x - S(x) & 0 < x < \pi \end{cases}$$

$$ODE: \begin{cases} S(x) = C_1 x + C_2 \\ S(0) = 1, S(\pi) = 0 \end{cases} \Rightarrow \begin{cases} S(0) = 1 \\ S(\pi) = 0 \end{cases} \Rightarrow \begin{cases} C_2 = 1 \\ C_1 \pi + C_2 = 0 \end{cases} \Rightarrow C_1 = -\frac{1}{\pi}$$

$$\Rightarrow S(x) = -\frac{1}{\pi}x + 1$$

PDE: Look for  $V(x, t) = X(x)\bar{T}(t)$

$$X\ddot{\bar{T}} - X''\bar{T} = 0 \Rightarrow \frac{\ddot{\bar{T}}}{\bar{T}} = \frac{X''}{X} = \lambda \stackrel{\lambda = -\mu^2}{\Rightarrow} \begin{cases} X'' = -\mu^2 X \\ \frac{\ddot{\bar{T}}}{\bar{T}} = -\mu^2 \end{cases}$$

$$\text{The First ODE: } \begin{cases} X'' = -\mu^2 X \\ X(0) = X(\pi) = 0 \end{cases} \Rightarrow \begin{cases} X(x) = A \cos \mu x + B \sin \mu x \\ X(0) = X(\pi) = 0 \end{cases}$$

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B \sin \mu x$$

$$X(\pi) = 0 \Rightarrow B \sin \mu \pi = 0 \Rightarrow \sin \mu \pi = 0 \Rightarrow \mu \pi = n\pi \Rightarrow \mu = n$$

$$n = 1, 2, \dots$$

$$S_0 \quad X_n(x) = \overset{=1}{B_n} \sin nx, \quad n=1, 2, \dots$$

and

$$T_n(t) = C_n e^{-\mu^2 t} = C_n e^{-n^2 t}, \quad n=1, 2, \dots$$

Hence

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin nx$$

Using the I.C.

$$v(x,0) = -\frac{1}{\pi}x - S(x) = -\frac{1}{\pi}x - \left(-\frac{1}{\pi}x + 1\right) = -1$$

$$\Rightarrow \sum_{n=1}^{\infty} C_n \sin nx = -1$$

$$\Rightarrow C_n = \frac{2}{\pi} \int_0^{\pi} (-1) \sin nx dx = \frac{2}{n\pi} \cos nx \Big|_0^{\pi} = \frac{2}{n\pi} ((-1)^n - 1)$$

Thus

$$u(x,t) = v(x,t) + S(x)$$

$$\Rightarrow u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n} e^{-n^2 t} \sin nx - \frac{1}{\pi}x + 1$$

Note that

$$u(x,t) \xrightarrow{t \rightarrow \infty} -\frac{1}{\pi}x + 1$$

