Matematik Chalmers

TMA683 Tillämpad matematik K2/Bt2

2019–04–24; KL 8:30-12:30

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Hjälpmedel: Endast tabell på backsidan av testen. Kalkylator ej tillåten.
Betygsgränser, **3**: 20–29p, **4**: 30–39p och **5**: 40–50p.
Lösningar/Granskning: Se kurshemsidan.

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1. Använd Laplacetransformer för att lösa differentialekvationen (8p)

$$y''(t) + 4y(t) = 8e^{2t}, \quad y(0) = 0, \quad y'(0) = 3.$$

2. (a) Bestäm Fourier sinus-serien med perioden 2π till funktionen $f(x) = \cos(x),\ 0 \le x \le \pi$. (4p)
(b) Bestäm interpolanten $\Pi_2 f \in \mathcal{P}^{(2)}[0,2]$ av funktionen $f(x) = x^3 + 1$ på en likformig partition $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ av intervallet $I = [0,2]$ och med de givna Lagrange basfunktioner $\mathcal{P}^{(2)}(0,2) = \text{Span}\{\lambda_0(x), \lambda_1(x), \lambda_2(x)\}$. (3p)

3. (a) Bestäm värdet på konstanten a så att funktionen $f(x) = ax^4 + x^3$ blir ortogonal mot alla konstanta funktioner på intervallet $[0,1]$. (4p)
(b) Bestäm $\|g\|_{L_2(0,2)}$ och $\|g\|_{L_\infty(0,2)}$ till funktionen

$$g(x) = \begin{cases} 1 - x, & 0 < x < 1, \\ e^{2x}, & 1 < x < 2. \end{cases}$$

(4p)

4. Betrakta den inhomogena värmeledningsekvationen (9p)

$$\begin{cases} \dot{u}(x,t) - u''(x,t) = 0, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 1,\ u(\pi,t) = 0, & t > 0, \\ u(x,0) = -\frac{1}{\pi}x & 0 < x < \pi. \end{cases}$$

Använd variabelseparationsmetoden för att bestämma $u(x,t)$.

5. Betrakta följande en-dimensionella vågekvation: (9p)

$$\begin{cases} \ddot{u} - u'' = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & \\ u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x) & 0 < x < 1. \end{cases}$$

Visa att den totala energin är konstant (konservering av energin). Dvs, visa att

$$\frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}\|u'(t)\|^2 = \text{konstant}.$$

6. Härled variationsformulering och finita element-formulering, samt berakna den styckvis linjära finita element-lösningen till randvärdesproblemet (9p)

$$\begin{cases} -u''(x) + 6u(x) = -2, & 0 < x < 1, \\ u(0) = u'(1) = 0, & \end{cases}$$

på en likformig partition $\mathcal{T}_h$ av intervallet $[0,1]$ med steglängd $h = \frac{1}{2}$.

LYCKA TILL!
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Table of Laplace Transforms and trigonomerty

$f(t)$	$F(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$e^{-at}f(t)$	$F(s+a)$
$f(t-T)\theta(t-T)$	$e^{-Ts}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
$\theta(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\frac{t}{2b}\sin bt$	$\frac{s}{(s^2+b^2)^2}$
$\frac{1}{2b^3}(\sin bt - bt\cos bt)$	$\frac{1}{(s^2+b^2)^2}$
$2\sin a\sin b \;= \cos(a-b)-\cos(a+b)$	
$2\sin a\cos b \;= \sin(a-b)+\sin(a+b)$	
$2\cos a\cos b \;= \cos(a-b)+\cos(a+b)$	

TMA683 Tillämpad matematik K2/Bt2, 2019–04–24; KL 8:30-12:30. Lösningar.

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1. Take the Laplace transform and consider the initial data $y(0) = 0,\ y'(0) = 3$,

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 8\frac{1}{s-2}$$

$$\implies \quad (s^2+4)Y(s) = 3 + \frac{8}{s-2} = \frac{3s+2}{s-2} \implies Y(s) = \frac{3s+2}{(s-2)(s^2+4)}$$

Now, using partial fractions, we have

$$\frac{3s+2}{(s-2)(s^2+4)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4} = \frac{(A+B)s^2 + (C-2B)s + 4A - 2C}{(s-2)(s^2+4)}$$

$$\implies \begin{cases} \text{A+B=0} \\ \text{C-2B=3} \\ \text{4A-2C=2} \end{cases} \implies A = 1,\ B = -1,\ C = 1$$

Hence

$$Y(s) = \frac{1}{s-2} + \frac{-s+1}{s^2+4} = \frac{1}{s-2} - \frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}$$

$$\implies y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{2t} - \cos 2t + \frac{1}{2}\sin 2t$$

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2. (a) For the Fourier sin series we have $a_n = 0,\ n = 0, 1, \dots$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos(x)\sin nx\ dx = \frac{1}{\pi}\int_0^{\pi}\sin(n-1)x\ dx + \frac{1}{\pi}\int_0^{\pi}\sin(n+1)x\ dx$$

$$\overset{n\neq 1}{=} \frac{1}{\pi}\Big(\frac{-1}{n-1}\cos(n-1)x\Big]_0^{\pi} + \frac{1}{\pi}\Big(\frac{-1}{n+1}\cos(n+1)x\Big]_0^{\pi}$$

$$= \frac{-1}{\pi(n-1)}\Big(\cos(n-1)\pi - \cos 0\Big) + \frac{-1}{\pi(n+1)}\Big(\cos(n+1)\pi - \cos 0\Big)$$

$$= \frac{(-1)^n+1}{\pi(n-1)} + \frac{(-1)^n+1}{\pi(n+1)} = \frac{2n\big((-1)^n+1\big)}{\pi(n^2-1)}$$

And for $n = 1$ we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\cos x \sin x\ dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\ dx = 0$$

Hence, for $0 \le x \le \pi$,

$$f(x) = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{n\big((-1)^n+1\big)}{n^2-1}\sin nx = \frac{4}{\pi}\Big(\frac{2}{2^2-1}\sin 2x + \frac{4}{4^2-1}\sin 4x + \cdots\Big)$$

$$= \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{2k}{(2k)^2-1}\sin 2k$$

(b) We have the interpolation points $x_0 = 0, x_1 = 1, x_2 = 2$, with corresponding function values $f(x_0) = 1, f(x_1) = 2, f(x_2) = 9$. The Lagrange polynomials are

$$\lambda_0 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{1}{2}(x-1)(x-2),$$

$$\lambda_1 = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = -x(x-2),$$

$$\lambda_2 = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_0)} = \frac{1}{2}x(x-1),$$

therefore we have

$$\pi_2 f(x) = f(x_0)\lambda_0(x) + f(x_1)\lambda_1(x) + f(x_2)\lambda_2(x) = 3x^2 - 2x + 1$$

3. (a) We should have

$$0 = \langle f, 1\rangle_{L_2(0,1)} = \int_0^1 (ax^4 + x^3)\cdot 1\ dx = \frac{a}{5} + \frac{1}{4}$$

that implies $a = -\frac{5}{4}$.
(b)

$$\|g\|_{L_2(0,2)} = \sqrt{\int_0^2 |g(x)|^2\ dx} = \sqrt{\int_0^1 (1-x)^2\ dx + \int_1^2 e^{4x}\ dx} = \sqrt{\frac{1}{3} + \frac{e^8 - e^4}{4}}$$

$$\|g\|_{L_\infty(0,2)} = \max_{x\in(0,2)} |g(x)| = \max(\sup_{x\in(0,1))} |1-x|, \sup_{x\in(1,2)} |e^{2x}|) = \max(1, e^4) = e^4$$

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4. We look for the solution as $u(x,t) = v(x,t) + s(x)$. Putting the solution in the PDE, we have

$$\begin{cases} \dot v(x,t) - v''(x,t) - s''(x) = 0, & 0 < x < \pi, \quad t > 0, \\ v(0,t) + s(0) = 1,\ v(\pi,t) + s(\pi) = 0, & t > 0, \\ v(x,0) + s(x) = -\frac{1}{\pi}x & 0 < x < \pi. \end{cases}$$

So we need to solve an ODE and a PDE:

$$\begin{cases} s''(x) = 0, & 0 < x < \pi, \\ s(0) = 1,\ s(\pi) = 0. \end{cases}$$

$$\begin{cases} \dot v(x,t) - v''(x,t) = 0, & 0 < x < \pi, \quad t > 0, \\ v(0,t) = 0,\ v(\pi,t) = 0, & t > 0, \\ v(x,0) = -\frac{1}{\pi}x - s(x) & 0 < x < \pi. \end{cases}$$

First we solve the ODE:

$$\begin{cases} s(x) = C_1 x + C_2 \\ s(0) = 1,\ s(\pi) = 0 \end{cases} \overset{s(0)=1, s(\pi)=0}{\Rightarrow} \begin{cases} C_2 = 1 \\ \pi C_1 + C_2 = 0 \ \Rightarrow C_1 = -\frac{1}{\pi} \end{cases} \Rightarrow\ s(x) = -\frac{1}{\pi}x + 1$$

Now, we solve the PDE. We look for the solution $v(x,t) = X(x)T(t)$. Then

$$X\dot T - X''T = 0 \implies \frac{\dot T}{T}(t) = \frac{X''}{X}(x) = \lambda \overset{\lambda=-\mu^2}{\implies} \begin{cases} X''(x) = -\mu^2 X(x) \\ \dot T(t) = -\mu^2 T(t) \end{cases}$$

For the first equation, considering the homogeneous boundary conditions, we have

$$\begin{cases} X''(x) = -\mu^2 X(x) \\ X(0) = X(\pi) = 0 \end{cases} \implies \begin{cases} X(x) = A\cos\mu x + B\sin\mu x \\ X(0) = X(\pi) = 0 \end{cases}$$

$X(0) = 0\ \Rightarrow\ A = 0\ \Rightarrow\ X(x) = B\sin\mu x$

$X(\pi) = 0\ \Rightarrow\ B\sin\mu\pi = 0 \overset{B\neq 0}{\implies} \sin\mu\pi = 0\ \Rightarrow\ \mu = n,\ n = 1, 2, \cdots$

So $X_n(x) = B_n \sin nx$ and for the second equation: $T_n(t) = C_n e^{-\mu^2 t} = C_n e^{-n^2 t}$.
Hence, by superposition principle, solution is

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2 t} \sin nx$$

Now, using the initial condition

$$v(x,0) = -\frac{1}{\pi}x - s(x) = -1$$

we have

$$v(x,0) = \sum_{n=1}^{\infty} C_n \sin nx = -1$$

and therefore

$$C_n = \frac{2}{\pi}\int_0^\pi (-1)\sin nx dx = \frac{2}{n\pi}\cos nx\Big]_0^\pi = \frac{2}{n\pi}\big((-1)^n - 1\big)$$

Hence

$$u(x,t) = v(x,t) + s(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}e^{-n^2 t}\sin nx - \frac{1}{\pi}x + 1$$

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5. See the book, Theorem 9.5.

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6. Define function space

$$V = \{v \mid v, v' \in L_2(0,1),\ v(0) = 0\}$$

Now, multiply the differential equation by a test function $v \in V$, then integrate over $(0,1)$ and integrate by parts:

$$-\int_0^1 u''v\ dx + 6\int_0^1 uv\ dx = -\int_0^1 2v\ dx$$

$$\Longrightarrow\ -\underbrace{u'(1)}_{=0}v(1) + u'(0)\underbrace{v(0)}_{=0} + \int_0^1 u'v'\ dx + 6\int_0^1 uv\ dx = -\int_0^1 2v\ dx$$

Hence the variational formulation (VF) is:

Find $u \in V$, such that

$$\int_0^1 u'v'\ dx + 6\int_0^1 uv\ dx = -2\int_0^1 v\ dx, \qquad \forall v \in V$$

Consider a uniform partition with constant mesh size $h = \frac{1}{2}$. Then the finite element space is:

$$V_h = \{v \mid v \text{ is continuous piecewise linear on } \mathcal{T}_h,\ v(0) = 0\} = span\{\varphi_1, \varphi_2\}$$

and note that φ_2 is a half hat function. Then, the finite element cG(1) method is:

Find $U \in V_h$, such that

$$\int_0^1 U'\chi'\ dx + 6\int_0^1 U\chi\ dx = -2\int_0^1 \chi\ dx, \qquad \forall \chi \in V_h$$

We note that $U(x) = \xi_1\varphi_1(x) + \xi_2\varphi_2(x)$. Now, to find ξ_1, ξ_2 we put $U(x)$ in the finite element method and set $\chi = \varphi_i,\ i = 1,2$:

$$\sum_{j=1}^{2}\Big(\int_0^1 \varphi_j'(x)\varphi_i'(x)\ dx\Big)\xi_j + 6\sum_{j=1}^{2}\Big(\int_0^1 \varphi_j(x)\varphi_i(x)\ dx\Big)\xi_j$$

$$= -2\int_0^1 \varphi_i(x)\ dx, \quad i = 1,2$$

that is the linear system of equations

$$(S + 6M)\xi = b$$

where, $\xi = \{\xi_j\}_{j=1}^2$ is the vector of unknowns, $S = \{\int_0^1 \varphi_j'\varphi_i'\ dx\}_{i,j=1}^2$ is the stiffness matrix, $M = \{\int_0^1 \varphi_j\varphi_i\ dx\}_{i,j=1}^2$ is the Mass matrix, and $F = \{F_i\}_{i=1}^2$ is the load vector with

$$F_i = -2\int_0^1 \varphi_i\ dx$$

Therefore

$$S = \frac{1}{h}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad M = \frac{h}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix},$$

and with $h = \frac{1}{2}$ we have

$$\left(2\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}$$

Hence $\xi = [-\frac{5}{21} - \frac{2}{7}]$.