Matematik Chalmers

**TMA683 Tillämpad matematik K2/Bt2**

**2019–08–22; KL 14:00-18:00**

Telefon: Johannes Borgqvist: 031-772 5325; Examinator: Fardin Saedpanah 031-772 3515
Hjlpmedel: Endast tabell p backsidan av testen. Kalkylator ej tillten.
Betygsgrnser, **3**: 2029p, **4**: 3039p och **5**: 4050p.
Lsningar/Granskning: Se kurshemsidan.

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**1.** Anvnd Laplacetransformer fr att lsa integro-differentialekvationen ($\theta$ r Heaviside funktionen) (6p)

$$y(t) + 2\int_0^t y(\tau)\,d\tau = \theta(t) - \theta(t-1), \quad y(0) = 0.$$

**2.** Lt $V = \mathbb{R}^3$ and $v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. Fr vilka vrden p $t$ kommer $y = \begin{bmatrix} -4 \\ 3 \\ t \end{bmatrix}$ att finnas i $\text{Span}\{v_1, v_2, v_3\}$ ? (7p)

**3.** (a) Bestm Fourierserien till $2\pi$-periodiska funktionen $f(x) = x$, $0 \le x \le 2\pi$. (5p)

(b) Anvnd resultatet i (a) till att berkna summan $S = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. (3p)

**4.** Betrakta den inhomogena vrmeledningsekvationen (10p)

$$\begin{cases} \dot{u}(x,t) - u''(x,t) = 4, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0,\ u(1,t) = 5, & \qquad\qquad\quad t > 0, \\ u(x,0) = -2x^2 & 0 < x < 1. \end{cases}$$

Anvnd variabelseparationsmetoden fr att bestmma $u(x,t)$.

**5.** Betrakta randvrdesproblemet (9p)

$$\text{(BVP)} \qquad \begin{cases} -\Big(a(x)u'(x)\Big)' = f, & 0 < x < 1, \\ u(0) = u(1) = 0, & . \end{cases}$$

dr $f \in \mathcal{C}(0,1)$ och $a \in \mathcal{C}^1(0,1)$. Ange en variationsformulering (VF) fr (BVP) och visa att

$$\text{(BVP)} \iff \text{(VF) och } u \in \mathcal{C}^2(0,1).$$

**6.** Betrakta begynnelsevrdesproblemet

$$\begin{cases} -u'' + 2u' + 3u = 4, & 0 < x < 1, \\ u(0) = 0, \quad u'(1) = 0. \end{cases}$$

(a) Hrled *variationsformulering.* (3p)

(b) Hrled cG(1) finita element furmulering (kontinuerliga stykvis linjra polynomer). Hrleda det linjra ekvationssystemet i formen $S\xi + 2C\xi + 3M\xi = F$. Berkna styvhetsmatris $S$ (Stiffness matrix) och lastvektor $F$ (Load vector). Ej konvektionsmatrisen $C$ (Convection matrix) och ej massmatris $M$ (Mass matrix). (7p)

(OBS! Anvnd likformig partition med steglngd $h$ och $\mathcal{T}_h:\ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$.)

LYCKA TILL! `\FS`

**Table of Laplace Transforms and trigonomerty**

| $f(t)$ | $F(s)$ |
|---|---|
| $af(t)+bg(t)$ | $aF(s)+bG(s)$ |
| $tf(t)$ | $-F'(s)$ |
| $t^n f(t)$ | $(-1)^n F^{(n)}(s)$ |
| $e^{-at}f(t)$ | $F(s+a)$ |
| $f(t-T)\theta(t-T)$ | $e^{-Ts}F(s)$ |
| $f'(t)$ | $sF(s)-f(0)$ |
| $f''(t)$ | $s^2F(s)-sf(0)-f'(0)$ |
| $f^{(n)}(t)$ | $s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$ |
| $\int_0^t f(\tau)\,d\tau$ | $\frac{F(s)}{s}$ |
| $\theta(t)$ | $\frac{1}{s}$ |
| $\frac{t^n}{n!}$ | $\frac{1}{s^{n+1}}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ |
| $\cosh at$ | $\frac{s}{s^2-a^2}$ |
| $\sinh at$ | $\frac{a}{s^2-a^2}$ |
| $\cos bt$ | $\frac{s}{s^2+b^2}$ |
| $\sin bt$ | $\frac{b}{s^2+b^2}$ |
| $\frac{t}{2b}\sin bt$ | $\frac{s}{(s^2+b^2)^2}$ |
| $\frac{1}{2b^3}(\sin bt - bt\cos bt)$ | $\frac{1}{(s^2+b^2)^2}$ |
| | |
| $2\sin a\sin b \ = \cos(a-b)-\cos(a+b)$ | |
| $2\sin a\cos b \ = \sin(a-b)+\sin(a+b)$ | |
| $2\cos a\cos b \ = \cos(a-b)+\cos(a+b)$ | |

**TMA683 Tillämpad matematik K2/Bt2, 2019–08–22; KL 14:00-18:00. Lösningar.**

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**1.** Take the Laplace transform

$$Y(s)+\frac{2}{s}Y(s)=\frac{1}{s}-\frac{e^{-s}}{s}$$

$$\Longrightarrow \frac{s+2}{s}Y(s)=\frac{1}{s}-\frac{e^{-s}}{s} \qquad \Longrightarrow \quad Y(s)=\frac{1}{s+2}-\frac{e^{-s}}{s+2}$$

Hence

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=e^{-2t}\theta(t)-e^{-2(t-1)}\theta(t-1).$$

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**2.** We have $y\in \text{Span}\{v_1,v_2,v_3\}$, if there are real numbers $a_1,a_2,a_3\in\mathbb{R}$ such that

$$y=a_1v_1+a_2v_2+a_3v_3$$

so

$$a_1\begin{bmatrix}1\\-1\\-2\end{bmatrix}+a_2\begin{bmatrix}5\\-4\\-7\end{bmatrix}+a_3\begin{bmatrix}-3\\1\\0\end{bmatrix}=\begin{bmatrix}-4\\3\\t\end{bmatrix}$$

That is, this linear system should have a non-zero solution:

$$\begin{cases}a_1+5a_2-3a_3=-4\\-a_1-4a_2+\ a_3=3\\-2a_1-7a_2 \qquad =t\end{cases} \underset{E_3+2E_1\to E_3}{\overset{E_2+E_1\to E_2}{\Longrightarrow}} \begin{cases}a_1+5a_2-3a_3=-4\\ \quad a_2-2a_3=-1\\ \quad 3a_2-6a_3=t-8\end{cases}$$

$$\overset{E_3-3E_2\to E_3}{\Longrightarrow} \begin{cases}a_1+5a_2-3a_3=-4\\ \quad a_2-2a_3=-1\\ \quad 0a_2+0a_3=t-5\end{cases} \Longrightarrow \quad t=5$$

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**3.** (a) $f$ is not an odd/even function, so we need to compute all Fourier coefficients.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\ dx=\frac{1}{\pi}\int_0^{2\pi}f(x)\ dx=\frac{1}{\pi}\int_0^{2\pi}x\ dx=\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}=2\pi$$

For $n=1,2,\dots,$

$$\begin{aligned}a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\ dx=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\ dx=\frac{1}{\pi}\int_0^{2\pi}x\cos nx\ dx\\&=\{u=x,\quad dv=\cos nx\ dx\}\\&=\frac{1}{n\pi}\underbrace{\left[x\sin nx\right]_0^{2\pi}}_{=0}-\frac{1}{n\pi}\int_0^{2\pi}\sin nx\ dx=\frac{1}{n^2\pi}\cos nx\Big]_0^{2\pi}=0\end{aligned}$$

And, for $n=1,2,\dots,$ we have

$$\begin{aligned}b_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\ dx=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\ dx=\frac{1}{\pi}\int_0^{2\pi}x\sin nx\ dx\\&=\{u=x,\quad dv=\sin nx\ dx\}\\&=-\frac{1}{n\pi}\underbrace{\left[x\cos nx\right]_0^{2\pi}}_{=2\pi}+\frac{1}{n\pi}\int_0^{2\pi}\cos nx\ dx=-\frac{2}{n}+\frac{1}{n^2\pi}\underbrace{\sin nx\Big]_0^{2\pi}}_{=0}=-\frac{2}{n}\end{aligned}$$

Hence

$$f(x) = \pi + \sum_{n=1}^{\infty} \frac{-2}{n} \sin nx = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$
$$= \pi - 2\Big(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5} + \cdots\Big)$$

(b) At $x = \frac{\pi}{2}$

$$\frac{\pi}{2} = f(\frac{\pi}{2}) = \pi - 2\Big(\sin\frac{\pi}{2} + \frac{\sin 2\frac{\pi}{2}}{2} + \frac{\sin 3\frac{\pi}{2}}{3} + \frac{\sin 4\frac{\pi}{2}}{4} + \frac{\sin 5\frac{\pi}{2}}{5} + \cdots\Big)$$
$$= \pi - 2\Big(1 - \frac{1}{3} + \frac{1}{5} - \cdots\Big) = \pi - 2\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$
$$\Longrightarrow \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$$

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**4.** We look for the solution as $u(x,t) = v(x,t) + s(x)$. Putting the solution in the PDE, we have

$$\begin{cases} \dot{v}(x,t) - v''(x,t) - s''(x) = 4, & 0 < x < 1, \quad t > 0, \\ v(0,t) + s(0) = 0,\ v(1,t) + s(1) = 5, & t > 0, \\ v(x,0) + s(x) = -2x^2 & 0 < x < 1. \end{cases}$$

So we need to solve an ODE and a PDE:

$$\begin{cases} -s''(x) = 4, & 0 < x < 1, \\ s(0) = 0,\ s(1) = 5. \end{cases}$$

$$\begin{cases} \dot{v}(x,t) - v''(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ v(0,t) = 0,\ v(1,t) = 0, & t > 0, \\ v(x,0) = -2x^2 - s(x) & 0 < x < 1. \end{cases}$$

First we solve the ODE:

$$\begin{cases} s(x) = -2x^2 + C_1 x + C_2 \\ s(0) = 0,\ s(1) = 5 \end{cases} \underset{s(1)=5}{\overset{s(0)=0}{\Longrightarrow}} \begin{cases} C_2 = 0 \\ -2 + C_1 + C_2 = 5 \ \Rightarrow C_1 = 7 \end{cases} \Rightarrow s(x) = -2x^2 + 7x$$

Now, we solve the homogeneous PDE, that is

$$\begin{cases} \dot{v}(x,t) - v''(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ v(0,t) = 0,\ v(1,t) = 0, & t > 0, \\ v(x,0) = -2x^2 - s(x) = -7x & 0 < x < 1. \end{cases}$$

We look for the solution $v(x,t) = X(x)T(t)$. Then

$$X\dot{T} - X''T = 0 \implies \frac{\dot{T}}{T}(t) = \frac{X''}{X}(x) = \lambda \overset{\lambda=-\mu^2}{\Longrightarrow} \begin{cases} X''(x) = -\mu^2 X(x) \\ \dot{T}(t) = -\mu^2 T(t) \end{cases}$$

For the first equation, considering the homogeneous boundary conditions, we have

$$\begin{cases} X''(x) = -\mu^2 X(x) \\ X(0) = X(1) = 0 \end{cases} \Longrightarrow \begin{cases} X(x) = A\cos\mu x + B\sin\mu x \\ X(0) = X(1) = 0 \end{cases}$$

$X(0) = 0 \ \Rightarrow \ A = 0 \ \Rightarrow \ X(x) = B\sin\mu x$

$X(1) = 0 \ \Rightarrow \ B\sin\mu = 0 \ \overset{B\neq 0}{\Longrightarrow} \ \sin\mu = 0 \ \Rightarrow \ \mu = n\pi,\ n = 1, 2, \cdots$

So $X_n(x) = B_n \sin n\pi x$ and for the second equation: $T_n(t) = C_n e^{-\mu^2 t} = C_n e^{-n^2\pi^2 t}$.

Hence, by superposition principle, solution is

$$v(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 t} \sin n\pi x$$

Now, using the initial condition $v(x,0) = -7x$, we have

$$v(x,0) = \sum_{n=1}^{\infty} C_n \sin n\pi x = -7x$$

and therefore

$$\begin{aligned}
C_n &= \frac{2}{1}\int_0^1 (-7x)\sin n\pi x\ dx = -14\int_0^1 x\sin n\pi x\ dx \\
&= \{u = x, \quad dv = \sin n\pi x\ dx\} \\
&= \frac{14}{n\pi}\underbrace{\big[x\cos n\pi x\big]_0^1}_{=(-1)^n} - \frac{14}{n\pi}\int_0^1 \cos n\pi x\ dx = \frac{14}{n\pi}(-1)^n - \frac{14}{n^2\pi^2}\underbrace{\big[\sin n\pi x\big]_0^1}_{=0} = \frac{14}{n\pi}(-1)^n
\end{aligned}$$

Hence

$$u(x,t) = v(x,t) + s(x) = \frac{14}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin n\pi x - 2x^2 + 7x$$

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**5.** See the book, Theorem 3.10.

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**6.** (a) Define function space

$$V = \{v \mid v, v' \in L_2(0,1),\ v(0) = 0\}.$$

Now, multiply the differential equation by a test function $v \in V$, then integrate over $(0,1)$ and integrate by parts:

$$\begin{aligned}
&-\int_0^1 u''v\ dx + 2\int_0^1 u'v\ dx + 3\int_0^1 uv\ dx = \int_0^1 4v\ dx \\
\Longrightarrow\ &-\underbrace{u'(1)}_{=0}v(1) + u'(0)\underbrace{v(0)}_{=0} + \int_0^1 u'v'\ dx + 2\int_0^1 u'v\ dx + 3\int_0^1 uv\ dx = 4\int_0^1 v\ dx
\end{aligned}$$

Hence the variational formulation (VF) is:

Find $u \in V,$ such that

$$\int_0^1 u'v'\ dx + 2\int_0^1 u'v\ dx + 3\int_0^1 uv\ dx = 4\int_0^1 v\ dx, \qquad \forall v \in V$$

(b) Consider a uniform partition with constant mesh size $h$:

$$\mathcal{T}_h:\ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$$

Then the finite element space is

$$V_h = \{v \mid v \text{ is continuous piecewise linear on } \mathcal{T}_h,\ v(0) = 0\} = span\{\varphi_1, \ldots, \varphi_m, \varphi_{m+1}\}$$

Then, the finite element method (FEM) is:

Find $U \in V_h,$ such that

$$\int_0^1 U'\chi'\ dx + 2\int_0^1 U'\chi\ dx + 3\int_0^1 U\chi\ dx = 4\int_0^1 \chi\ dx, \qquad \forall \chi \in V_h$$

We note that $U \in V_h$ and $U(x) = \sum_{j=1}^{m+1} \xi_j \varphi_j(x)$. Now, to find $m+1$ unknowns $\xi_1, \ldots, \xi_{m+1}$, we put $U(x)$ in the finite element method and set $\chi = \varphi_i,\ i = 1, \cdots, m+1$:

$$\int_0^1 \Big( \sum_{j=1}^{m+1} \xi_j \varphi_j'(x) \Big) \varphi_i'(x)\ dx + 2 \int_0^1 \Big( \sum_{j=1}^{m+1} \xi_j \varphi_j'(x) \Big) \varphi_i(x)\ dx$$

$$+ 3 \int_0^1 \Big( \sum_{j=1}^{m+1} \xi_j \varphi_j(x) \Big) \varphi_i(x)\ dx = 4 \int_0^1 \varphi_i(x)\ dx, \quad i = 1, \ldots, m+1$$

$$\Longrightarrow \sum_{j=1}^{m+1} \underbrace{\Big( \int_0^1 \varphi_j'(x) \varphi_i'(x)\ dx \Big)}_{=S_{i,j}} \xi_j + 2 \sum_{j=1}^{m+1} \underbrace{\Big( \int_0^1 \varphi_j'(x) \varphi_i(x)\ dx \Big)}_{=C_{i,j}} \xi_j$$

$$+ 3 \sum_{j=1}^{m+1} \underbrace{\Big( \int_0^1 \varphi_j(x) \varphi_i(x)\ dx \Big)}_{=M_{i,j}} \xi_j = \underbrace{4 \int_0^1 \varphi_i(x)\ dx}_{=F_i}, \quad i = 1, \ldots, m+1$$

that is the linear system of equations

$$S\xi + 2C\xi + 3M\xi = F$$

For the stiffness matrix we have (note that $\varphi_{m+1}$ is a half hat function):
for $i = 1, \ldots, m$,

$$S_{i,i} = \int_0^1 \varphi_i' \varphi_i'\ dx = \int_{x_{i-1}}^{x_i} \big(\frac{1}{h}\big)^2\ dx + \int_{x_i}^{x_{i+1}} \big(-\frac{1}{h}\big)^2\ dx = \frac{2}{h}$$

$$S_{i,i+1} = S_{i+1,i} = \int_0^1 \varphi_i' \varphi_{i+1}'\ dx = \int_{x_i}^{x_{i+1}} \big(-\frac{1}{h}\big)\big(\frac{1}{h}\big)\ dx = -\frac{1}{h}$$

and

$$S_{m+1,m+1} = \int_0^1 \varphi_{m+1}' \varphi_{m+1}'\ dx = \int_{x_m}^{x_{m+1}} \big(\frac{1}{h}\big)^2\ dx = \frac{1}{h}$$

That is

$$S = \frac{1}{h} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}_{(m+1)\times(m+1)}$$

For the load vector, for $i = 1, \ldots, m$,

$$F_i = \int_0^1 \varphi_i\ dx = \int_{x_{i-1}}^{x_{i+1}} \varphi_i\ dx = h$$

and

$$F_{m+1} = \int_0^1 \varphi_{m+1}\ dx = \int_{x_m}^{x_{m+1}} \varphi_i\ dx = \frac{h}{2}$$

That is

$$F = h \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1/2 \end{bmatrix}_{(m+1)\times 1}$$