Matematik Chalmers

TMA683 Tillämpad matematik K2/Bt2

2019–01–19; KL 8:30-12:30

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Hjälpmedel: Endast tabell på backsidan av testen. Kalkylator ej tillåten.
Betygsgränser, **3**: 20–29p, **4**: 30–39p och **5**: 40–50p.
Lösningar/Granskning: Se kurshemsidan.

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1. Använd Laplacetransformer för att lösa integro-differentialekvationen (8p)

$$y'(t) + 2y(t) + 2\int_0^t y(\tau)\,d\tau = 1 + e^{-t}, \quad y(0) = 1.$$

2. (a) Bestäm Fourierserien till 2π-periodiska funktionen (4p)

$$f(x) = \begin{cases} 1 & 0 < \mathrm{x} < \pi, \\ \text{-}1 & \text{-}\pi < x < 0. \end{cases}$$

(b) Använd resultatet i (a) till att beräkna summan (3p)

$$S = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}$$

3. (a) Bestäm L_2-projektionen av $f(x) = 3x^2 - 4x^3$ i $\mathcal{P}^{(1)}(0,1)$. (4p)

(b) Visa att $\{1, t-1, t^2\}$ är linjärt oberoende i $\mathcal{P}^{(2)}(\mathbb{R})$. (4p)

4. Betrakta vågekvationen (9p)

$$\begin{cases} \ddot{u} - u'' = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & \qquad\qquad\quad t > 0, \\ u(x,0) = \sin 2\pi x, & 0 < x < 1, \\ \dot{u}(x,0) = \sin \pi x, & 0 < x < 1 \end{cases}$$

Använd variabelseparationsmetoden för att bestämma $u(x,t)$.

5. Betrakta värmeledningsekvationen för (8p)

$$\begin{cases} \dot{u}(x,t) - u''(x,t) = f(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & \qquad\qquad\quad t > 0, \\ u(x,0) = u_0(x) & 0 < x < 1. \end{cases}$$

Bevisa följande stabilitetsrelationen:

$$\|u(\cdot,t)\| \le \|u_0\| + \int_0^t \|f(\cdot,s)\|\,ds.$$

6. Betrakta begynnelsevärdesproblemet

$$\begin{cases} -u'' + u = 1, \quad 0 < x < 1, \\ u(0) = 2, \quad u'(1) = 3. \end{cases}$$

(a) Härled *variationsformulering.* (3p)

(b) Härled cG(1) finita element furmulering (kontinuerliga stykvis linjära polynomer). Härleda det linjära ekvationssystemet i formen $S\xi + M\xi = F$, och beräkna styvhetsmatris S (Stiffness matrix). Ej lastvektor F (Load vector) och ej massmatris M (Mass matrix). (7p)

(OBS! Använd likformig partition med steglängd h och $\mathcal{T}_h :\ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1.$)

LYCKA TILL!
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Table of Laplace Transforms and trigonomerty

$f(t)$	$F(s)$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$e^{-at}f(t)$	$F(s+a)$
$f(t-T)\theta(t-T)$	$e^{-Ts}F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-\sum_{k=1}^{n} s^{n-k}f^{(k-1)}(0)$
$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
$\theta(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\frac{t}{2b}\sin bt$	$\frac{s}{(s^2+b^2)^2}$
$\frac{1}{2b^3}(\sin bt - bt\cos bt)$	$\frac{1}{(s^2+b^2)^2}$
$2\sin a\sin b \ = \cos(a-b)-\cos(a+b)$	
$2\sin a\cos b \ = \sin(a-b)+\sin(a+b)$	
$2\cos a\cos b \ = \cos(a-b)+\cos(a+b)$	

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1. Take the Laplace transform and consider the initial data $y(0) = 1$,

$$sY(s) - y(0) + 2Y(s) + 2\frac{Y(s)}{s} = \frac{1}{s} + \frac{1}{s+1}$$

$$\Longrightarrow \quad \frac{s^2+2s+2}{s}Y(s) = \frac{1}{s} + \frac{1}{s+1} + 1 \Longrightarrow Y(s) = \frac{s^2+3s+1}{(s+1)(s^2+2s+2)}$$

Now, using partial fractions, we have

$$\frac{s^2+3s+1}{(s+1)(s^2+2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2} = \frac{(A+B)s^2+(2A+B+C)s+2A+C}{(s+1)(s^2+2s+2)}$$

$$\Longrightarrow \begin{cases} \text{A+B=1} \\ \text{2A+B+C=3} \\ \text{2A+C=1} \end{cases} \Longrightarrow A = -1,\ B = 2,\ C = 3$$

Hence

$$Y(s) = \frac{-1}{s+1} + \frac{2s+3}{s^2+2s+2} = \frac{-1}{s+1} + \frac{(2s+2)+1}{(s^2+2s+1)+1} = \frac{-1}{s+1} + 2\frac{(s+1)}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

$$\Longrightarrow y(t) = -e^{-t} + 2e^{-t}\cos t + e^{-t}\sin t = (-1 + 2\cos t + \sin t)e^{-t}$$

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2. (a) f is an odd function, so $a_n = 0$ for $n = 0, 1, 2, \dots$, and for $n = 1, 2, \dots$,

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\ dx = -\frac{2}{n\pi}(\cos n\pi - \cos 0) = \frac{2}{n\pi}\big(1 - (-1)^n\big)$$

so

$$f(x) = \frac{2}{\pi}\sum_{n=1}^\infty \frac{1-(-1)^n}{n}\sin nx = \frac{4}{\pi}\Big(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\Big)$$

(b) At $x = \frac{\pi}{2}$

$$1 = f(\frac{\pi}{2}) = \frac{4}{\pi}\Big(\sin\frac{\pi}{2} + \frac{\sin 3\frac{\pi}{2}}{3} + \frac{\sin 5\frac{\pi}{2}}{5} + \cdots\Big) = \frac{4}{\pi}\Big(1 - \frac{1}{3} + \frac{1}{5} + \cdots\Big)$$

$$= \frac{4}{\pi}\sum_{k=1}^\infty \frac{(-1)^{k-1}}{2k-1}$$

$$\Longrightarrow \sum_{k=1}^\infty \frac{(-1)^{k-1}}{2k-1} = \frac{\pi}{4}$$

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3. (a) $Pf \in \mathcal{P}^{(1)}(0,1)$, so $Pf(x) = \xi_0 + \xi_1 x,\ x \in (0,1)$. To find the unknowns $\xi_0,\ \xi_1$ we need two equations. By the definition of L_2-projection:

$$\int_0^1 f(x)x^i\ dx = \int_0^1 (Pf)(x)x^i\ dx, \quad i = 0, 1$$

so

$$\begin{cases} \int_0^1 (3x^2 - 4x^3)\ dx = \int_0^1 (\xi_0 + \xi_1 x)\ dx \\ \\ \int_0^1 (3x^2 - 4x^3)x\ dx = \int_0^1 (\xi_0 + \xi_1 x)x\ dx \end{cases} \Longrightarrow \begin{cases} \xi_0 + \frac{1}{2}\xi_1 = 0 \\ \\ \frac{1}{2}\xi_0 + \frac{1}{3}\xi_1 = -\frac{1}{20} \end{cases} \Longrightarrow \xi_0 = \frac{3}{10}, \quad \xi_1 = -\frac{3}{5}$$

Hence $\quad (Pf)(x) = \frac{3}{10} - \frac{3}{5}x$.

(b) Consider a linear combination of $1, t-1, t^2$ as

$$\alpha_1 + \alpha_2(t-1) + \alpha_3 t^2 = 0$$

Then for $t = 0, 1, -1$ we have

$$\Longrightarrow \begin{cases} \alpha_1 - \alpha_2 = 0 \\ \alpha_1 + \alpha_3 = 0 \\ \alpha_1 - 2\alpha_2 + \alpha_3 = 0 \end{cases} \quad \Longrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

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4. We look for $u(x,t) = X(x)T(t)$: from the differential equation we have

$$X\ddot{T} - X''T = 0 \Longrightarrow \frac{\ddot{T}}{T}(t) = \frac{X''}{X}(x) = \lambda \overset{\lambda=-\mu^2}{\Longrightarrow} \begin{cases} X''(x) = -\mu^2 X(x) \\ \ddot{T}(t) = -\mu^2 T(t) \end{cases}$$

For the first ODE, considering the homogeneous boundary conditions, we have

$$\begin{cases} X''(x) = -\mu^2 X(x) \\ X(0) = X(1) = 0 \end{cases} \Longrightarrow \begin{cases} X(x) = A\cos\mu x + B\sin\mu x \\ X(0) = X(1) = 0 \end{cases}$$

$X(0) = 0 \;\Rightarrow\; A = 0 \;\Rightarrow\; X(x) = B\sin\mu x$

$X(1) = 0 \;\Rightarrow\; B\sin\mu = 0 \overset{B\neq 0}{\Longrightarrow} \sin\mu = 0 \;\Rightarrow\; \mu = n\pi,\ n = 1, 2, \cdot$

So $X_n(x) = B_n \sin n\pi x$ and for the second ODE: $T_n(t) = C_n \cos n\pi t + D_n \sin n\pi t$. Hence, by superposition principle, solution is

$$u(x,t) = \sum_{n=1}^{\infty} \Big(C_n \cos n\pi t + D_n \sin n\pi t \Big) \sin n\pi x$$

and

$$\dot{u}(x,t) = \sum_{n=1}^{\infty} \Big(-n\pi C_n \sin n\pi t + n\pi D_n \cos n\pi t \Big) \sin n\pi x$$

Now, using the initial values, we have

$$\begin{cases} u(x,0) = \sum_{n=1}^{\infty} C_n \sin n\pi x = \sin 2\pi x \\ \\ \dot{u}(x,0) = \sum_{n=1}^{\infty} n\pi D_n \sin n\pi x = \sin \pi x \end{cases} \Longrightarrow \begin{cases} C_2 = 1, \quad C_n = 0,\ n = 1, 3, 4, \cdots \\ \\ D_1 = \frac{1}{\pi}, \quad D_n = 0,\ n = 2, 3, 4, \cdots \end{cases}$$

Hence

$$u(x,t) = \cos 2\pi t \sin 2\pi x + \frac{1}{\pi} \sin \pi t \sin \pi x$$

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5. See the book, Theorem 9.1.

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6. (a) Define function spaces

$$V = \{v \mid v, v' \in L_2(0,1),\ v(0) = 2\}, \quad \tilde{V} = \{v \mid v, v' \in L_2(0,1),\ v(0) = 0\}$$

Now, multiply the differential equation by a test function $v \in \tilde{V}$, then integrate over $(0,1)$ and integrate by parts:

$$-\int_0^1 u''v\ dx + \int_0^1 uv\ dx = \int_0^1 v\ dx$$

$$\Longrightarrow -\underbrace{u'(1)}_{=3} v(1) + u'(0)\underbrace{v(0)}_{=0} + \int_0^1 u'v'\ dx + \int_0^1 uv\ dx = \int_0^1 v\ dx$$

Hence the variational formulation (VF) is:

$$\text{Find } u \in V, \text{ such that}$$

$$\int_0^1 u'v'\ dx + \int_0^1 uv\ dx = \int_0^1 v\ dx + 3v(1), \qquad \forall v \in \tilde{V}$$

(b) Consider a uniform partition with constant mesh size h:

$$\mathcal{T}_h:\ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$$

Then the finite element spaces are

$$V_h = \{v \mid v \text{ is continuous piecewise linear on } \mathcal{T}_h,\ v(0) = 2\} = span\{\varphi_0, \varphi_1, \dots, \varphi_m, \varphi_{m+1}\}$$

$$\tilde{V}_h = \{v \mid v \text{ is continuous piecewise linear on } \mathcal{T}_h,\ v(0) = 0\} = span\{\varphi_1, \dots, \varphi_m, \varphi_{m+1}\}$$

Then, the finite element method (FEM) is:

$$\text{Find } U \in V_h, \text{ such that}$$
$$\int_0^1 U'\chi'\ dx + \int_0^1 U\chi\ dx = \int_0^1 \chi\ dx + 3\chi(1), \qquad \forall \chi \in \tilde{V}_h$$

We note that $U \in V_h$ and

$$U(x) = \xi_0\varphi_0(x) + \xi_1\varphi_1(x) + \cdots + \xi_m\varphi_m(x) + \xi_{m+1}\varphi_{m+1}(x)$$
$$\overset{U(0)=2}{\Longrightarrow} \ 2 = U(0) = \xi_0 \cdot 1 + \xi_1 \cdot 0 + \cdots + \xi_m \cdot 0 + \xi_{m+1} \cdot 0 = \xi_0$$
$$\Longrightarrow\ U(x) = 2\varphi_0(x) + \sum_{j=1}^{m+1} \xi_j\varphi_j(x)$$

Now, to find $m+1$ unknowns $\xi_1, \dots, \xi_{m+1}$, we put $U(x)$ in the finite element method and set $\chi = \varphi_i,\ i = 1, \cdots, m+1$:

$$\int_0^1 \Big(2\varphi_0'(x) + \sum_{j=1}^{m+1} \xi_j\varphi_j'(x)\Big)\varphi_i'(x)\ dx + \int_0^1 \Big(2\varphi_0(x) + \sum_{j=1}^{m+1} \xi_j\varphi_j(x)\Big)\varphi_i(x)\ dx$$
$$= \int_0^1 \varphi_i(x)\ dx + 3\varphi_i(x_{m+1}), \quad i = 1, \dots, m+1$$
$$\Longrightarrow \sum_{j=1}^{m+1} \Big(\int_0^1 \varphi_j'(x)\varphi_i'(x)\ dx\Big)\xi_j + \sum_{j=1}^{m+1} \Big(\int_0^1 \varphi_j(x)\varphi_i(x)\ dx\Big)\xi_j$$
$$= \int_0^1 \varphi_i(x)\ dx + 3\varphi_i(x_{m+1}) - 2\int_0^1 \varphi_0'(x)\varphi_i'(x)\ dx - 2\int_0^1 \varphi_0(x)\varphi_i(x)\ dx, \quad i = 1, \dots, m+1$$

that is the linear system of equations

$$S\xi + M\xi = F$$

where, $\xi = \{\xi_j\}_{j=1}^{m+1}$ is the vector of unknowns, $S = \{\int_0^1 \varphi_j'\varphi_i'\ dx\}_{i,j=1}^{m+1}$ is the stiffness matrix, $M = \{\int_0^1 \varphi_j\varphi_i\ dx\}_{i,j=1}^{m+1}$ is the Mass matrix, and $F = \{F_i\}_{i=1}^{m+1}$ is the load vector with

$$F_i = \int_0^1 \varphi_i\ dx + 3\varphi_i(x_{m+1}) - 2\int_0^1 \varphi_0'\varphi_i'\ dx - 2\int_0^1 \varphi_0\varphi_i\ dx$$

For the stiffness matrix we have (note that φ_{m+1} is a half hat function):
for $i = 1, \dots, m$,

$$S_{i,i} = \int_0^1 \varphi_i'\varphi_i'\ dx = \int_{x_{i-1}}^{x_i} \big(\frac{1}{h}\big)^2\ dx + \int_{x_i}^{x_{i+1}} \big(-\frac{1}{h}\big)^2\ dx = \frac{2}{h}$$

$$S_{i,i+1} = S_{i+1,i} = \int_0^1 \varphi_i'\varphi_{i+1}'\ dx = \int_{x_i}^{x_{i+1}} \big(-\frac{1}{h}\big)\big(\frac{1}{h}\big)\ dx = -\frac{1}{h}$$

and

$$S_{m+1,m+1} = \int_0^1 \varphi_{m+1}'\varphi_{m+1}'\ dx = \int_{x_m}^{x_{m+1}} \big(\frac{1}{h}\big)^2\ dx = \frac{1}{h}$$

That is

$$S = \frac{1}{h} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}_{(m+1)\times(m+1)}$$

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