TMA690 Recommended Problems - Week 1.

Problem 1.1. Problem 11.1 from the book.

Problem 1.2. Read the last paragraph on page 163 and Example 11.5. Then solve Problem 11.2 from the book.

Problem 1.3. Determine whether the differential equation

$$\frac{\partial^2 u}{\partial x_1 x_2} + \frac{\partial^2 u}{\partial x_2 x_1} + 2\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial u}{\partial x_2} + u = 0$$

is elliptic, parabolic or hyperbolic.

Problem 1.4. Let V be an inner product space over $\mathbb{R}$. Show the following identities.

(a) Parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2), \quad u, v \in V.$$

(b) Pythagoras' Theorem: if $u, v \in V$ with $u \perp v$, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

(c) The polarization identity:

$$(u, v) = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2).$$

Problem 1.5. Prove that if $(V, \|\cdot\|)$ is a normed space, then $\big|\, \|v\| - \|w\| \,\big| \leq \|v - w\|$ for every $v, w \in V$. Deduce that the norm is a continuous function from V to $\mathbb{R}$.

Problem 1.6. Let $a > b$ and $V = C[a, b]$ be the vector space of real valued continuous functions over $\mathbb{R}$ with the usual addition and scalar multiplication. Let

$$(f, g) = \int_a^b f(x)g(x)\,\mathrm{d}x, \quad f, g \in V.$$

Show that $(\cdot\,,\cdot)$ is well-defined and defines an inner product on V.

Problem 1.7. Show that $C[0, 2]$ with the above inner product is not a Hilbert space. Hint: let $\{f_n\}$ be the sequence in $C([0, 2])$ defined by:

$$f_n(x) = \begin{cases} 1 & \text{if } x \leq 1 - 1/n \\ n - nx & \text{if } 1 - 1/n < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$

Show $\{f_n\}$ is Cauchy but it cannot converge to a continuous function.

Problem 1.8. Let $(V, (\cdot\,,\cdot))$ as in Problem 1.6.

(a) Show that $F(v) = \int_a^b v(x)\mathrm{d}x$, $v \in V$, is a continuous linear functional on V. Determine its norm.

(b) Show that $G(v) = v(a)$, $v \in V$, is a linear functional on V but it is not continuous.

(c) Show that

$$a(u, v) = \int_a^b u(x)v(x)(1 + x^2)\,\mathrm{d}x, \quad u, v \in V,$$

is a bounded, symmetric, positive definite, bilinear form on V.

Problem 1.9. Show that $V = C[a,b]$ with norm

$$\|v\| = \sup_{x\in[a,b]} |v(x)|$$

is a Banach space.

Problem 1.10. Let ℓ^2 be the set of sequences $x = (x_n) \subset \mathbb{R}$ of numbers satisfying $\sum_{n=1}^{\infty} x_n^2 < \infty$. Show that ℓ^2 is a vector space with the operations $\lambda x + \mu y := (\lambda x_n + \mu y_n)$, $\lambda, \mu \in \mathbb{R}$, and that for $x, y \in \ell^2$ the series

$$\sum_{n=1}^{\infty} x_n y_n$$

converges in $\mathbb{R}$. Thus we can define a function $(\cdot\,,\cdot) : \ell^2 \times \ell^2 \to \mathbb{R}$ by $(x\,,y) = \sum_{n=1}^{\infty} x_n y_n$. Show that $(\cdot\,,\cdot)$ is an inner product on ℓ^2.

Problem 1.11. (Optional) Show that ℓ^2 is a Hilbert space.

Problem 1.12. Let V, W be normed spaces over $\mathbb{R}$ and let $B : V \to W$ be a bounded linear operator with norm defined by

$$\|B\| = \sup_{v\in V, v\neq 0} \frac{\|Bv\|_W}{\|v\|_V}.$$

Show that

$$\|B\| = \sup_{v\in V, \|v\|=1} \|Bv\| = \inf\{C \in \mathbb{R} : \|Bv\|_W \leq C\|v\|_V \text{ for all } v \in V\}.$$

Problem 1.13. Let V, W be normed spaces over $\mathbb{R}$ and let $B : V \to W$ be a linear operator. Show that B is a bounded linear operator iff B is continuous.

Problem 1.14. Let V, W be normed spaces over $\mathbb{R}$. Show that the space of bounded linear operators $\mathcal{B}(V, W)$ is a Banach space if W is a Banach space.