TMA690 Recommended Problems - Week 2.

**Problem 2.1.** Show that on $C[a,b]$, the $\|\cdot\|_{L_2}$-norm and the supremum norm are not equivalent.

**Problem 2.2.** Show that the bilinear form

$$a(u,v) = \int_0^1 u(x)v(x)(1+x^2)\,\mathrm{d}x, \quad u,v \in V,$$

is coercive on $L^2(0,1)$. Show, however, that

$$a(u,v) = \int_0^1 u(x)v(x)x^2\,\mathrm{d}x, \quad u,v \in V,$$

is not coercive, only positive definite. Are the norms induced by the two bilinear forms equivalent?

**Problem 2.3.** Use the definition of a set having Lebesgue measure 0 from class when solving the following problems.

(a) Show that if $A \subset \mathbb{R}^d$ is countable, then $m(A) = 0$.

(b) Show that for $A = \{(x,0) : x \in \mathbb{R}\} \subset \mathbb{R}^2$ it holds that $m(A) = 0$.

**Problem 2.4.** Find the supremum and essential supremum on $\Omega = (-1,1) \times (-1,1)$ of the function defined by $v(x,y) = 1$ if $y \neq 0$ and $v(x,0) = \frac{1}{x^2}$ $(x \neq 0)$, $v(0,0) = 1$.

**Problem 2.5.** Suppose that $V, W$ are Banach spaces and $A \subset V$ is a dense subspace of $V$.[1] Suppose that $T : A \to W$ is a bounded linear operator. Show that there is a unique bounded linear extension $\tilde{T} : V \to W$ of $T$ to the whole of $V$ such that $\|T\|_{\mathcal{B}(A,W)} = \|\tilde{T}\|_{\mathcal{B}(V,W)}$.

**Problem 2.6.** Give a rigorous argument proving the fact that if $v \in C^1(\bar{\Omega})$, $\Omega \subset \mathbb{R}^d$, then the classical derivative $\frac{\partial v}{\partial x_j}$ coincides with its weak derivative.

**Problem 2.7.** Problems A.1, A.2, A.4-A.11 from the book.

[1] Hence $A$ is a normed space itself with the norm inherited from $V$.