TMA690 Recommended Problems - Week 3.

Problem 3.1. Let Ω be the open unit disc in $\mathbb{R}^2$. Consider the boundary value problem

$$-\nabla \cdot \big((1+x^2y^2)\nabla u\big) = \frac{1}{\sqrt[4]{x^2+y^2}}, \quad (x,y) \in \Omega;$$
$$u = 1 \quad \text{on } \Gamma.$$

Write down the weak formulation of the problem and prove that there is a unique weak solution.

Problem 3.2. Let $\Omega \subset \mathbb{R}^d$ be a domain, $F = (f_1, \dots, f_d)$ and $G = (g_1, \dots, g_d)$, where $f_i, g_i \in L^2(\Omega)$. Prove that

$$|(F,G)| := \left| \int_\Omega F \cdot G \, \mathrm{d}x \right| \leq \left(\int_\Omega \sum_{i=1}^d f_i^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega \sum_{i=1}^d g_i^2 \, \mathrm{d}x \right)^{1/2} := \|F\|_{L^2} \|G\|_{L2}.$$

Problem 3.3. From the book: Problems 3.1, 3.4-3.9, 3.13.