TMA690 Recommended Problems - Week 5.

Problem 4.1. Let $v : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Define

$$F(x,t) := \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} v(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, t > 0.$$

(a) Show that $\frac{\partial F}{\partial x}$ exists[1] for all $x \in \mathbb{R}$, $t > 0$, and

$$\frac{\partial F}{\partial x}(x,t) = -\int_{-\infty}^{\infty} \frac{x-y}{2t} e^{-\frac{(x-y)^2}{4t}} v(y)\,\mathrm{d}y.$$

Hint: Write

$$F(x,t) = e^{-\frac{x^2}{4t}} \int_{-\infty}^{\infty} e^{\frac{xy}{2t}} e^{-\frac{y^2}{4t}} v(y)\,\mathrm{d}y := e^{-\frac{x^2}{4t}} G(x,t)$$

and show that $\frac{\partial G}{\partial x}$ exists and derive a formula for $\frac{\partial G}{\partial x}$. Then apply the product rule.

(b) Show that $\frac{\partial^2 F}{\partial x^2}$ and derive a formula for $\frac{\partial^2 F}{\partial x^2}$.

Problem 4.2. Let $v : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Define

$$F(x,t) := \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} v(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, t > 0.$$

Show that $\frac{\partial F}{\partial t}$ exists[2] for all $x \in \mathbb{R}, t > 0$ and

$$\frac{\partial F}{\partial t}(x,t) = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \frac{(x-y)^2}{4t^2} v(y)\,\mathrm{d}y.$$

Hint: Consider first

$$G(x,r) = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4}r} v(y)\,\mathrm{d}y,$$

show that $\frac{\partial G}{\partial r}$ exists and derive a formula for $\frac{\partial G}{\partial r}$. Then notice that $F(x,t) = G(x, \frac{1}{t})$ and use the chain rule.

Problem 4.3. Let $v : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Let

$$u(x,t) := (4\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} v(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, t > 0. \tag{1}$$

Show that u is a classical solution to the heat equation $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$.

Problem 4.4. Let $v : \mathbb{R} \to \mathbb{R}$ be bounded and continuous. Show that the solution u of the heat equation in 1D, given by (1), satisfies

$$\|D^\alpha u\|_{C(\mathbb{R})} \le C t^{-5/2} \|v\|_{C(\mathbb{R})}, \quad \alpha = (1,2),$$

for some $C > 0$.

Problem 4.5. Write down a solution formula in polar coordinates for

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= g \text{ on } \Gamma, \end{aligned}$$

where $\Omega = \{x \in \mathbb{R}^2 : |x| < R\}$, f, g are appropriately smooth, $f(x) = h(|x|)$ for some $h : \mathbb{R} \to \mathbb{R}$ and $g = g(\varphi)$, $\varphi \in [0, 2\pi)$.

Problem 4.6. Let $U(x) = \frac{1}{4\pi|x|}$, $x \in \mathbb{R}^3 \setminus \{0\}$. Show that $(U, -\Delta\varphi) = \varphi(0)$ for all $\varphi \in C_0^2(\mathbb{R}^3)$.

[1] Actually, $\frac{\partial^n F}{\partial x^n}$ exists for all $n \in \mathbb{N}$ for $x \in \mathbb{R}, t > 0$ which can be shown by induction.

[2] In fact, $\frac{\partial^n F}{\partial t^n}$ exists for all $n \in \mathbb{N}$ for $x \in \mathbb{R}, t > 0$ which can be shown by induction.