TMA690 Recommended Problems - Week 6.

Problem 5.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary Γ. Show that if u is an appropriately smooth solution of the equation

$$\begin{aligned} -\Delta u + u^3 &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma, \end{aligned}$$

then $u = 0$.

Problem 5.2. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary Γ and let $T > 0$. Let $c = c(x)$ be continuous on $\bar{\Omega}$ and $c(x) > 0$ on Ω. Show that if u is appropriately smooth on $\bar{\Omega} \times [0, T]$ and $u_t - \Delta u + cu \leq 0$ on $\Omega \times (0, T)$, then

$$\max_{\bar{\Omega} \times [0,T]} u \leq \max(0, \max_{\Gamma_p} u),$$

where $\Gamma_p = (\Gamma \times [0, T]) \cup (\Omega \times \{0\}.)$
Hint: Modify the proof of Theorem 8.6 in the book.

Problem 5.3. Let u be a classical solution to the heat equation

$$\begin{aligned} \partial_t u - \partial_x^2 u = (x^2 - 1)u, \ &\text{in } (0,1) \times (0, \infty) \\ u(0,t) = u(1,t) = 0, \ &\text{for } t > 0 \\ u(x,0) = \sqrt{x(1-x)}, \ &\text{for } x \in [0,1]. \end{aligned}$$

(a) Show that $\|u(\cdot, t)\|^2 \leq \frac{1}{6}$ for $t \geq 0$.

(b) Show that $u(x,t) \leq \frac{1}{2}$ for $(x,t) \in [0,1] \times [0, \infty)$.

Problem 5.4. Solve the equation

$$e^x \partial_x u + \partial_y u = 0, \quad u(0,y) = y^4.$$

Problem 5.5. Use the method of descent to derive D'Alambert's formula (the solution formula for the pure initial value problem for the wave equation in one spatial dimension) from the solution formula for the pure initial value problem for the wave equation in two spatial dimensions.

Problem 5.6. Find the constant a such that the equation $(x^2 + x)\delta' = a\delta$, where δ is the Dirac delta distribution.