Problem 1.1. $\frac{\partial^2 u}{\partial x_1^2} + x_2 \frac{\partial^2 u}{\partial x_2^2} = f$

characteristic equation: $\xi_1^2 + x_2 \xi_2^2 = 0$.

- If $x_1 > 0 \Rightarrow$ No characteristic directions
- If $x_1 = 0 \Rightarrow (0,1)$ is a characteristic direction and $x_2 = c$ are characteristics.
- If $x_1 < 0 \Rightarrow \xi_1^2 = -x_1 \xi_2^2 \Rightarrow (\pm\sqrt{-x_1}\xi, \xi) = \xi(\pm\sqrt{-x_1}, 1)$ $(\xi \in \mathbb{R}, \xi \neq 0)$ are characteristic directions at (x_1, x_2) for any $\xi \in \mathbb{R}, \xi \neq 0$.

Characteristics: $F(x_1, x_2) = c$ such that

$$\frac{\partial F}{\partial x_1} = \pm\sqrt{-x_1} \Rightarrow F(x_1, x_2) = \mp \frac{2}{3}(-x_1)^{3/2} + C(x_2)$$

$$\frac{\partial F}{\partial x_2} = 1 \Rightarrow C'(x_2) = 1 \Rightarrow C(x_2) = x_2 + c_1$$

$$\Rightarrow F(x_1, x_2) = \mp\frac{2}{3}(-x_1)^{3/2} + x_2 + c_1$$

$\Rightarrow$ characteristics: $\mp(-x_1)^{3/2} + \frac{3}{2}x_2 = D \quad (D \in \mathbb{R})$

Problem 1.2. We rewrite the system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \quad \text{as}$$

using $\mathcal{U} = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\frac{\partial \mathcal{U}}{\partial x} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix}$, $\frac{\partial \mathcal{U}}{\partial y} = \begin{pmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{pmatrix}$,

$$A_1 \frac{\partial \mathcal{U}}{\partial x} + A_2 \frac{\partial \mathcal{U}}{\partial y} = 0, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

characteristic equation:

$$\det(\xi_1 A_1 + \xi_2 A_2) = \det\begin{bmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{bmatrix} = \xi_1^2 + \xi_2^2 = 0$$

$\Rightarrow$ No characteristic directions.

Problem 1.14. We need to show that $\mathcal{B}(V,W)$ is complete.

Let $(T_n) \subset \mathcal{B}(V,W)$ be a Cauchy sequence in $\mathcal{B}(V,W)$:

$(\forall \varepsilon>0)(\exists N)\ m,n \geq N \Rightarrow \|T_n - T_m\| < \varepsilon$; i.e. $\|T_m - T_n\| \to 0$ as $m,n\to\infty$.

We have to prove that T_n is convergent. Fix $v \in V$. Then

$$\|T_n v - T_m v\| = \|(T_n - T_m)v\| \leq \|T_m - T_n\|\,\|v\| \to 0 \quad \text{as } m,n \to \infty.$$

Hence $(T_n v) \subset W$ is Cauchy hence converges as W is a Banach space (complete). Let $w \in W$ be $\lim T_n v = w$ and define a mapping $T: V \to W$ by $Tv = \lim_{n\to\infty} T_n v$.

① T is linear. $T(\lambda v + \mu u) = \lim T_n(\lambda v + \mu u) = \lim \lambda T_n v + \mu T_n u$

$= \lambda T v + \mu T u$

② $\big|\|T_n\| - \|T_m\|\big| \leq \|T_m - T_n\| \to 0$ as $m,n \to \infty$.

Hence $\{\|T_n\|\} \subset \mathbb{R}$ is Cauchy, therefore bounded.

$\Rightarrow \exists C>0 \qquad \|T_n\| \leq C \qquad n \in \mathbb{N}$.

③ $\|Tu\| = \|\lim T_n u\| = \lim \|T_n u\| \leq \limsup \|T_n\| \cdot \|u\| \leq C\|u\|$.

$\Rightarrow$ T is a bounded operator.

④ To show $T_n \to T$ in the norm of $\mathcal{B}(V,W)$.

Let $\varepsilon > 0$. Then $\exists N)\ m,n \geq N \quad \|T_m - T_n\| < \frac{\varepsilon}{2}$. Fix $n \geq N$.

Then

$$\|T_n x - Tx\| = \lim_{m\to\infty} \|T_n x - T_m x\| = \lim_{m\to\infty} \|(T_m - T_n)x\|$$

$$\leq \limsup \|T_m - T_n\|\,\|x\| \leq \frac{\varepsilon}{2}\|x\|$$

$$\Rightarrow n \geq N \Rightarrow \frac{\|T_n x - Tx\|}{\|x\|} \leq \frac{\varepsilon}{2} \Rightarrow \|T_n - T\| \leq \frac{\varepsilon}{2} < \varepsilon. \qquad \square$$

Problem 1.11. See homepage.