Problem set 2.

(2.1) (a) $\|f\|_{L^2}^2 = \int_a^b |f(x)|^2 dx \le \sup_{[a,b]} |f(x)|^2 \cdot (b-a)$

$$= \Big(\underbrace{\sup |f(x)|}_{\|f\|_C}\Big)^2 (b-a)$$

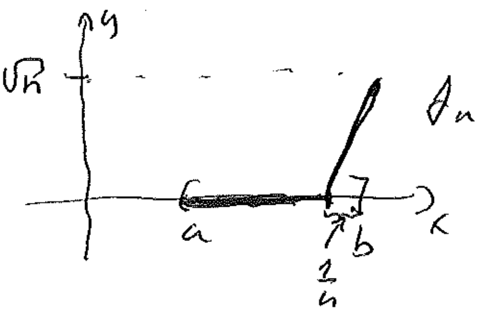
$$\Rightarrow \|f\|_{L^2} \le \sqrt{b-a}\, \|f\|_C .$$

But not the other way around!

We argue by contradiction. Suppose $\exists \alpha > 0$:

$$(*) \quad \|f\|_C \le \alpha \|f\| \qquad \forall f \in C[a,b].$$

Consider f_n:



$$\|f_n\|_C = \sup_{x \in [a,b]} |f_n(x)| = \sqrt{n}$$

$$\|f_n\|_{L^2}^2 = \int_a^b f_n^2 dx = \int_{b-1/n}^b f_n^2 dx \le \int_{b-1/n}^b n\, dx = \frac{1}{n} \cdot n = 1.$$

Thus, by $(*)$, we must have:

$$\sqrt{n} \le \alpha \qquad \text{for all } n = 1, 2, \dots$$

which is impossible no matter what α is. ↯.

Hence $(*)$ cannot hold for any $\alpha > 0$ and thus $\|\cdot\|_{L^2}$ and $\|\cdot\|_C$ are not equivalent.

(2.2) (a)

$$|a(v,v)| = \int_0^1 v^2(1+x^2)\,dx \geq \int_0^1 v^2\,dx = \|v\|_{L^2}^2 .$$

Thus a is coercive with coercivity constant $\alpha = 1$.

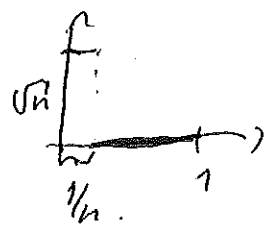
(b) We first show that $a(u,v) = \int_0^1 x^2 u v\,dx$ is positive definite.

- $|a(v,v)| = \int_0^1 x^2 v^2\,dx \geq 0$
- Suppose that $a(v,v) = \int_0^1 x^2 v^2\,dx = 0$.

As $x^2 v^2 \geq 0 \Rightarrow x^2 v^2 = 0$ a.e. $\Rightarrow v = 0$ a.e.
Thus a is positive definite.
But a is not coercive: Suppose, by contradiction, that a is coercive: there is $\alpha > 0$ such that

$$(**) \quad |a(v,v)| \geq \alpha \|v\|^2 \qquad \forall v \in L^2 .$$

Let f_n:



We have that $\|f_n\|^2 = \int_0^1 f_n^2\,dx = \int_0^{1/n} n\,dx = 1$.

Furthermore,

$$a(f_n,f_n) = \int_0^1 f_n^2 x^2\,dx = \int_0^{1/n} n x^2\,dx = n\left[\frac{x^3}{3}\right]_0^{1/n} = \frac{1}{3n^2}$$

Thus, by $(**)$, we must have

$$\frac{1}{3n^2} \geq \alpha \qquad \text{for all } n = 1,2,\ldots$$

$$\Rightarrow \quad n^2 \leq \frac{1}{3\alpha} \qquad \text{for all } n = 1,2,\ldots$$

This is impossible no matter what α is. Thus $(**)$ cannot hold for any α and a is not coercive.

Problem (2-3) .

(a). Let $A \subset \mathbb{R}^d$ countable:

$$A = \{a_1, \dots a_n, \dots\} \qquad a_n = (a_n^1, \dots, a_n^d)$$

Let $\varepsilon > 0$. Define a countable number of rectangles R_n by

$$R_n := \left(a_n^1 - \frac{\varepsilon}{2^{n+1}}, a_n^1 + \frac{\varepsilon}{2^{n+1}}\right) \times \left(a_n^2 - \frac{1}{2}, a_n^2 + \frac{1}{2}\right) \times \dots \times \left(a_n^d - \frac{1}{2}, a_n^d + \frac{1}{2}\right)$$

Then: $m(R_n) = \frac{\varepsilon}{2^n} \cdot 1 \cdot 1 \dots 1 = \frac{\varepsilon}{2^n}$. We have

$$A \subset \bigcup_{n=1}^{\infty} R_n \quad \text{and} \quad \sum_{n=1}^{\infty} m(R_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \varepsilon$$

Thus A can be covered by a countable number of rectangles with total "volume" less than equal to ε, for any $\varepsilon >$. Thus, by definition, $m(A) = 0$.

(b) Let $\varepsilon > 0$. Consider the rectangles

$$A_n := (n, n+2) \times \left(\frac{-\varepsilon}{2^{n+4}}, \frac{\varepsilon}{2^{n+4}}\right)$$

Then $A \subset \bigcup_{n=-\infty}^{\infty} A_n \qquad (A = \{(x,0) : x \in \mathbb{R}\} \subset \mathbb{R}^2)$

$$m(A_n) = 2 \cdot \frac{\varepsilon}{2^{n+3}} = \frac{\varepsilon}{2^{n+2}}$$

$$\sum_{n=-\infty}^{\infty} m(A_n) = m(A_0) + \sum_{n=1}^{\infty} m(A_n) + \sum_{n=-\infty}^{1} m(A_n)$$

$$\leq \frac{\varepsilon}{4} + 2\varepsilon \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} = \frac{\varepsilon}{4} + \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{\varepsilon}{4} + \frac{\varepsilon}{3} < \varepsilon.$$

Thus, by definition, $m(A) = 0$.

(2.4) Note that $v = 1$ except for the line segments $[-1,0) \times \{0\} \cup (0,1] \times \{0\}$. This has two dimensional Lebesgue measure 0. Hence $v = 1$ a.e. on Ω. Thus

$$\operatorname*{ess\,sup}_{\Omega} v = 1$$

However v takes on arbitrary large values on these line segments ($\frac{1}{x^2} \to \infty$ as $x \to 0$) and hence $\sup_{\Omega} v = +\infty$.

(2.5) Let $x \in V$. As $A \subset V$ is dense there is $(x_n) \subset A$ such that $x_n \to x$. Then (x_n) is Cauchy and hence

$$\|Tx_n - Tx_m\|_W = \|T(x_n - x_m)\|_W \le \|T\|_{\mathcal{B}(A,W)} \|x_n - x_m\|_V \to 0 \quad \text{as } n,m \to \infty$$

$\Rightarrow$ (Tx_n) is Cauchy in W. As W is complete, there is $y \in W$ such that $Tx_n \to y$. Define $\tilde{T}x = y = \lim_{n \to \infty} Tx_n$.

- $\tilde{T}$ is well defined: If $(x_n), (z_n) \subset A$ s.t. $x_n \to x$, $z_n \to x$ with $Tx_n \to y_1$ and $Tz_n \to y_2$

$\Rightarrow$

$$\|y_1 - y_2\|_W = \lim_{n \to \infty} \|Tx_n - Tz_n\|_W \le \|T\|_{\mathcal{B}(A,W)} \|x_n - z_n\| \to 0 \quad \text{as } n \to \infty.$$

$\Rightarrow y_1 = y_2$. Thus the definition of $\tilde{T}$ is independent of the choice of the approximating sequence.

- $\tilde{T}$ is an extension: Let $x \in A$. Take $x_n = x$ $\forall n$. Then $x_n \to x$. Therefore $\tilde{T}x = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Tx = Tx$.

- $\tilde{T}$ is linear: $\tilde{T}(\lambda x + \mu y) = \lambda \tilde{T}x + \mu \tilde{T}y$.

Let $x_n \to x$, $y_n \to y$ then $\lambda x_n + \mu y_n \to \lambda x + \mu y$

$$\Rightarrow \tilde{T}(\lambda x + \mu y) = \lim T(\lambda x_n + \mu y_n) = \lim(\lambda T x_n + \mu T y_n)$$
$$= \lambda \lim Tx_n + \mu \lim Ty_n = \lambda \tilde{T}x + \mu \tilde{T}y.$$

• $\tilde{T}$ is bounded

$$\|\tilde{T}x\|_W = \|\lim Tx_n\|_W = \lim \|Tx_n\|_W \le \limsup \|T\|_{\mathcal{B}(A,W)} \|x_n\|_V$$

$$= \|T\|_{\mathcal{B}(A,W)} \|x\|$$

$$\Rightarrow \frac{\|\tilde{T}x\|_W}{\|x\|_V} \le \|T\|_{\mathcal{B}(A,W)} \Rightarrow \|\tilde{T}\|_{\mathcal{B}(V,W)} = \sup_{x\in V} \frac{\|\tilde{T}x\|_W}{\|x\|_V} \le \|T\|_{\mathcal{B}(A,W)}$$

On the other hand:

$$\|T\|_{\mathcal{B}(A,W)} = \sup_{x\in A} \frac{\|Tx\|_W}{\|x\|_V} = \sup_{x\in A} \frac{\|\tilde{T}x\|_W}{\|x\|_V} \le \sup_{x\in V} \frac{\|\tilde{T}x\|_W}{\|x\|_V} = \|\tilde{T}\|_{\mathcal{B}(V,W)}$$

$$\Rightarrow \|T\|_{\mathcal{B}(A,W)} = \|\tilde{T}\|_{\mathcal{B}(V,W)}$$

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2.6 Let $v \in C^1(\bar{\Omega})$. It's weak derivatives, for $\phi \in C_0^1(\Omega)$,

$L(\phi) = -\int_\Omega v \frac{\partial \phi}{\partial x_j} dx = \int \frac{\partial v}{\partial x_j} \phi \, dx$, where we used "integration by parts"

as both $v, \phi \in C^1$. As $\frac{\partial v}{\partial x_j} \in C(\bar{\Omega}) \Rightarrow \frac{\partial v}{\partial x_j} \in L^2(\Omega)$. Thus

$L(\phi) = \left(\frac{\partial v}{\partial x_j}, \phi\right)$. ~~If where~~

$|L(\phi)| \le \|\frac{\partial v}{\partial x_j}\| \, \|\phi\| \Rightarrow L$ is a bounded (clearly linear) functional defined on a dense subset $C_0^1(\Omega) \subset L^2(\Omega)$. Thus, it has a unique bounded extension to L^2: $L(\phi) = (v_j, \phi)$ by the Riesz representation theorem. But then

$$(v_j, \phi) - \left(\frac{\partial v}{\partial x_j}, \phi\right) = \left(v_j - \frac{\partial v}{\partial x_j}, \phi\right) = 0 \quad \forall \phi \in C_0^1(\Omega).$$

As $C_0^1(\Omega)$ is dense in $L^2 \Rightarrow v_j = \frac{\partial v}{\partial x_j}$ a.e.

(A4) Let $(v_n) \subset H^1$ be a Cauchy sequence. Let $\varepsilon > 0$. (6)

Then there is $N > 0$ such that

$$n, m > N \Rightarrow \|v_n - v_m\|^2 + \|\nabla(v_n - v_m)\|^2 < \varepsilon^2$$

$$\Rightarrow \|v_n - v_m\| < \varepsilon \quad \text{and} \quad \|\nabla v_n - \nabla v_m\| < \varepsilon$$

As L^2 is complete, there is $v \in L^2$ such that $v_n \to v$ in L^2 and there is $w_k \in L^2$ such that $\frac{\partial v_n}{\partial x_k} \to w_k \in L^2$.

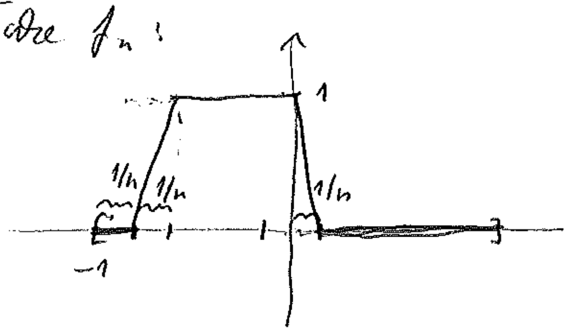
<u>To show</u>: $v \in H^1$ and $\frac{\partial v}{\partial x_k} = w_k$. We have, for $\phi \in C_0^1(\Omega)$,

$$\frac{\partial v}{\partial x_i}(\phi) = \int_\Omega v \frac{\partial \phi}{\partial x_k} dx = \lim_n \int v_n \frac{\partial \phi}{\partial x_i} dx \underset{v_n \in H^1}{=} \lim_n \int \frac{\partial v_n}{\partial x_k} \phi \, dx = \int w_k \phi \, dx$$

$\Rightarrow \frac{\partial v}{\partial x_k} \in L^2$; that is $v \in H^1$, and $\frac{\partial v}{\partial x_k} = w_k$.

(A5)

Take f_n:



$f_n \in C_0$

$$\|f_n - f\|_{L^2}^2 \le \frac{1}{n} + \frac{1}{n} + \frac{1}{n} = \frac{3}{n} \to 0 \quad \text{as } n \to \infty.$$

(A6) (a) $d = 1$: $\int_{-1}^{1} |x|^{2\lambda} dx < \infty \iff \lambda > -1/2$

$d = 2$: $\int_{B_1(0)} |x|^{2\lambda} dx = \int_0^{2\pi} \int_0^1 r^{2\lambda} \cdot r \, dr \, d\theta < \infty$ iff $\lambda > -1$

$d = 3$: $\int_{B_1(0)} |x|^{2\lambda} dx = \int_0^{\pi} \int_0^{2\pi} \int_0^1 r^{2\lambda} r^2 \sin\phi \, dr \, d\theta \, d\phi \qquad \lambda > -3/2$

(b) $d=1$: $\int_{-1}^{1} (\lambda |x|^{\lambda-1})^2 \, dx = \lambda^2 \int_0^1 |x|^{2\lambda-2} \, dx < \infty \iff \lambda > \frac{1}{2}$

$d=2$: $\int_{B_1(0)} |D|x|^\lambda|^2 \, dx = \int_0^{2\pi} \int_0^1 \lambda^2 |x|^{2\lambda-2} \cdot r \, dr \, d\alpha < \infty \iff \lambda > 0$

$d=3$: $\int_{B_1(0)} |D|x|^\lambda|^2 \, dx = \int_0^{\pi} \int_0^{2\pi} \int_0^1 \lambda^2 r^{2\lambda-2} r^2 \sin\phi \, dr \, d\alpha \, d\phi < \infty \iff \lambda > -\frac{1}{2}$.

A.7 ① $v \in L^2$: $\int_0^{2\pi} \int_0^{1/2} (\log(-\log r^2))^2 r \, dr \, d\theta < \infty$

as $(\log(-\log r^2)) r$ is bounded on $[0,1]$.

② $W^{1} \in L^2$. $Dv = (x, y) \cdot \frac{2}{+\log|x|^2 \cdot |x|^2}$

$|Dv|^2 = \frac{\cancel{4}}{(2\log|x|)^2 |x|^{\cancel{2}}}$

$\int_{B_{1/2}(0)} \frac{1}{(\log|x|)^2 |x|^2} \, dx = \int_0^{2\pi} \int_0^{1/2} \frac{1}{(\log r)^2 \cdot r^2} \cdot r \, dr \, d\alpha$

$= \int_0^{2\pi} \int_0^{1/2} \frac{1}{(\underbrace{\log r}_{s})^2} \cdot \underbrace{\frac{1}{r} \, dr}_{ds} \, d\alpha = 2\pi \int_{-\infty}^{\log 1/2} \frac{1}{s^2} \, ds < \infty.$

No, functions in $H^1(\Omega)$ are not necessarily bounded and continuous as this example shows.

A8 $H^1_0(\Omega)$ is a ~~closed~~ subspace of $H^1(\Omega)$.

Therefore, as $C^1_0(\Omega)$ is dense in $H^1_0(\Omega)$ it cannot be dense in $H^1(\Omega)$: sequences of functions from $C^1_0(\Omega)$ cannot have a limit (in the H^1 norm) outside of H^1_0 as H^1_0 is closed (and $C^1_0 \subset H^1_0$). Hence one cannot approximate functions in the H^1-norm that belongs to $H^1 \setminus H^1_0$.

A9

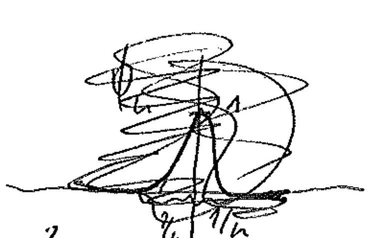
$$f'(\phi) = -\int_{-1}^{1} f\phi' dx = -\int_{-1}^{0} 0\,\phi' dx - \int_{0}^{1} x\,\phi' dx = 0 - \underbrace{[x\phi]_0^1}_{0 \text{ as } \phi(1)=0} + \int_0^1 1\,\phi\, dx$$

$$= \int_{-1}^{1} g(x)\phi(x)\,dx.$$

$$g'(\phi) = -\int_{-1}^{1} g\phi' dx = -\int_0^1 \phi'(x)\,dx = -[\phi]_0^1 = \underbrace{-\phi(1)}_{0} + \phi(0) = \phi(0).$$

$\Rightarrow$ As $g \in L^2 \Rightarrow f \in H^1$.

But g' is not continuous w.r.t. the L^2 norm. Let (ϕ_n) be such that $\phi_n(0) = 1$ and



$\|\phi_n\|_{L_2} \to 0$. If g' is a continuous linear functional on L^2, then there is $C>0$:

$$1 = \|g'(\phi_n)\| \le C\|\phi_n\| \to 0 \quad \text{as } n \to \infty$$

which is impossible.

Example: $\phi_n(x) := e^{-(nx)^2}$

(1.10) Let $f \in L^2$ and define a linear functional ~~from~~ $L_f(v) = (f, v)$ $v \in L^2$. Then for $v \in H_0^1$:

$$|L_f(v)| = \left|\int f v \, dx\right| \le \|f\| \, \|v\| \underset{\substack{\uparrow \\ \text{Poincaré}}}{\le} C\|f\| \, |v|_1$$

$$\Rightarrow \frac{|L_f(v)|}{|v|_1} \le C\|f\| \Rightarrow L_f \in (H_0^1)^* = H^{-1}.$$

$\Rightarrow L_2(\Omega) \subset H^{-1}$ in the sense that the linear functional L_f corresponding to $f \in L^2$ belongs to H^{-1}.

(1.11) Let $f(x) = \frac{1}{x}$. Then $f \notin L^2(0,1)$ as $\int_0^1 \frac{1}{x^2} dx = \infty$.

Let $v \in C_0^1$. ~~Then~~ and consider $L_f(v) = \int_0^1 f v \, dx$.

Then

$$|L_f(v)| = \int_0^1 \frac{1}{x} v \, dx = \Big| \underbrace{\left[\ln x \, v\right]_0^1}_{\substack{= 0 \\ v \equiv 0 \text{ on } (0,\varepsilon] \text{ for some } \varepsilon > 0.}} - \int_0^1 \ln x \, v' \, dx \Big|$$

$$= \left|\int_0^1 \ln x \, v' \, dx\right| \le \underbrace{\left(\int_0^1 \ln^2 x \, dx\right)^{1/2}}_{\wedge \atop C} \left(\int_0^1 v'^2 dx\right)^{1/2} = C|v|_1.$$

$\Rightarrow \frac{|L_f(v)|}{|v|_1} \le C$ for all $v \in C_0^1(0,1)$. As $C_0^1 \subset H_0^1$ dense $\Rightarrow \frac{|L_f(v)|}{|v|_1} \le C$ $\forall v \in H_0^1$

and also dense in L^2

$\Rightarrow L_f \in H^{-1}$ but $L_f \notin L^2$. ~~matter~~

$\int \ln^2 x \, dx = \int x \ln^2 x - \int x \, 2 \ln x \frac{1}{x} \, dx = \ldots$ well known.