solutions to the problems in the book.

Problem 3.1.

$$-\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\left(a_{jk}\frac{\partial u}{\partial x_k}\right) + a_0 u = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \Gamma$$

$a_0(x) \geq 0$, $\sum a_{jk}(x)\xi_j\xi_k \geq \kappa|\xi|^2$, $a_{jk} = a_{kj}$

Take $v \in C_0^1(\Omega)$. Then

$$\int_\Omega f v\,dx = -\sum_{j,k}\int_\Omega \frac{\partial}{\partial x_j}\left(a_{jk}\frac{\partial u}{\partial x_k}\right) v\,dx + \int_\Omega a_0 u v\,dx$$

$$= -\sum_{j,k}\int_\Gamma n_{x_j} a_{jk}\frac{\partial u}{\partial x_k} v\,ds + \sum_{j,k}\int a_{jk}\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_j}\,dx + \int_\Omega a_0 u v\,dx.$$

Weak formulation: Find $u \in H_0^1$ such that

$$\underbrace{\sum_{j,k=1}^{d}\int_\Omega a_{jk}\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_j}\,dx}_{(A\nabla u, \nabla v)} + \int_\Omega a_0 u v\,dx = \int f v\,dx \quad \forall v \in H_0^1(\Omega).$$

Set $a(u,v) = (A\nabla u, \nabla v) + (a_0 u, v)$

$L(v) = (f, v)$.

Let $V = H_0^1$ with $|\cdot|_1$.

- a is bilinear, symmetric ✓
- a is bounded:

$$|a(u,v)| \leq \|A\nabla u\|\,\|\nabla v\| \leq \max_{i,j=1}^{d}\|a_{ij}\|_{L^\infty(\Omega)}\|\nabla u\|\,\|\nabla v\|$$

$$= C|u|_1|v|_1$$

- a is coercive:

$$a(v,v) = \int\sum_{j,k} a_{jk}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_k}\,dx + \int_\Omega a_0 v^2\,dx$$

$$\geq \kappa\int|\nabla v|^2\,dx = \kappa|v|_1^2.$$

- L is bounded: $|L(v)| \leq \|f\|\|v\| \leq C\|f\|\,|v|_1$ by Poincaré.

$\Rightarrow$ By Riesz $\exists!\, u \in H_0^1 = V$ such that $a(u,v) = L(v)$ $\forall v \in H_0^1$.

Problem 3.4. Let $\phi = \frac{1}{2d}|x|^2$. Then $\Delta\phi = 1$.

Therefore
$$\int_\Omega v^2\,dx = \int_\Omega v^2 \Delta\phi\,dx = \int_\Gamma v^2 \nabla\phi\cdot n\,ds - 2\int_\Omega v\nabla v\cdot\nabla\phi\,dx$$

$$\Rightarrow \quad \|v\|^2 \le \int_\Gamma v^2|\nabla\phi|\,ds + 2\int_\Omega |v||\nabla v||\nabla\phi|\,dx.$$

If Ω is bounded then there is $M>0$ such that $\max_{\bar\Omega}|\nabla\phi| \le M$. Thus

$$\underbrace{\|v\|^2} \le M\int_\Gamma v^2\,ds + 2M\int_\Omega |v||\nabla v|\,dx.$$

~~Recall~~ $\cancel{ab = (\varepsilon a)\left(\frac{b}{\varepsilon}\right) \le \frac{\varepsilon^2 a^2}{2} + \frac{b^2}{2\varepsilon^2}}$

~~Choosing~~

$$\le M\|v\|^2_{L^2(\Gamma)} + \int_\Omega \frac{|v|^2}{2} + \frac{(2M|\nabla v|)^2}{2}\,dx$$

$$= \underbrace{M\|v\|^2_{L^2(\Gamma)} + \frac{1}{2}\|v\|^2 + 2M^2|v|_1^2}$$

$$\Rightarrow \quad \frac{1}{2}\|v\|^2 \le M\|v\|^2_{L^2(\Gamma)} + 2M^2|v|_1^2$$

$$\Rightarrow \quad \|v\|^2 \le C\left(\|v\|^2_{L^2(\Gamma)} + |v|_1^2\right).$$

$\square$

This is Friedrich's inequality.

Problem 3.5 Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ (3)

Then $v(x) = v(y) + \int_{y_1}^{x_1} D_1 v(s, x_2)\, ds + \int_{y_2}^{x_2} D_2 v(y_1, s)\, ds$

integrating $\int_0^1\int_0^1 dy$:

$$v(x) = \int_0^1\int_0^1 v(y)\, dy + \int_0^1 \int_{y_1}^{x_1} D_1 v(s, x_2)\, ds\, dy_1 + \int_0^1\int_0^1 \int_{y_2}^{x_2} D_2 v(y_1, s)\, ds\, dy$$

$\Rightarrow$

$$v(x)^2 \le 3\Big(\int_\Omega v(y)\, dy\Big)^2 + (\quad)^2 + (\quad)^2\Big)$$

$$\le 3\left[\Big(\int_\Omega v(y)\, dy\Big)^2 + \Big(\int_0^1\int_0^1 |D_1 v(s, x_2)|\, ds\, dy_1\Big)^2 + \Big(\int_0^1\int_0^1\int_0^1 |D_2 v(y_1, s)|\, ds\, dy\Big)^2\right]$$

$$= 3\left[\Big(\int_\Omega v(y)\, dy\Big)^2 + \Big(\int_\Omega 1 \cdot |D_1 v(y)|\, dy\Big)^2 + \Big(\int_\Omega 1 \cdot |D_2 v(y)|\, dy\Big)^2\right]$$

$$\le 3\Big(\Big(\int v(y)\, dy\Big)^2 + \int_\Omega 1^2 dy \int_\Omega D_1 v(y)^2\, dy + \int_\Omega 1^2 dy \int_\Omega D_2 v(y)^2\, dy\Big)$$

$$= 3\Big(\Big(\int_\Omega v\, dy\Big)^2 + \int_\Omega |\nabla v|^2\, dy\Big)$$

$$\Rightarrow \int_0^1\int_0^1 v(x)^2\, dx \le 3\left[\Big(\int_\Omega v\, dy\Big)^2 + \Big(\int_\Omega |\nabla v|^2\, dy\Big)\right]$$

$$\Rightarrow \|v\|^2 \le C\Big(\Big(\int_\Omega v\, dy\Big)^2 + \|\nabla v\|^2\Big).$$

<u>Problem 3.6</u>

$$-\Delta u = f \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial n} + u = g \quad \text{on } \Gamma$$

Take $v \in C^1$ and multiply the equation, then integrate:

$$\int_\Omega f v\,dx = \int_\Omega -\Delta u\, v\,dx = -\int_\Gamma \underbrace{\frac{\partial u}{\partial n}}_{g-u} v\,ds + \int_\Omega \nabla u\cdot\nabla v\,dx$$

$$= \int_\Gamma (u-g)v\,ds + \int_\Omega \nabla u\cdot\nabla v\,dx.$$

<u>Weak formulation:</u> Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u\cdot\nabla v\,dx + \int_\Gamma uv\,ds = \int_\Omega fv\,dx + \int_\Gamma gv\,ds, \quad \forall v \in H^1(\Omega).$$

Let $V = H^1(\Omega)$. Define

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v\,dx + \int_\Gamma uv\,ds$$

$$L(v) = \int_\Omega fv\,dx + \int_\Gamma gv\,ds$$

- a is bilinear, symmetric ✓
- a is bounded:

$$|a(u,v)| \le \|\nabla u\|\,\|\nabla v\| + \|u\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)}$$

$$\le \|u\|_1\|v\|_1 + C^2\|u\|_1\|v\|_1 \quad \text{by the trace inequality.}$$

$$= (1+C^2)\|u\|_1\|v\|_1.$$

- a is coercive:

$$a(v,v) = |v|_1^2 + \|v\|_{L^2(\Gamma)}^2 \ge |v|_1^2 + \tfrac12\|v\|_{L^2(\Gamma)}^2 = \tfrac12|v|_1^2 + \tfrac12\left(|v|_1^2 + \|v\|_{L^2(\Gamma)}^2\right)$$

$$\text{(Friedrich's ineq.)} \ge \tfrac12|v|_1^2 + C\|v\|_{L^2(\Omega)}^2 \ge \min(\tfrac12, C)\left(|v|_1^2 + \|v\|^2\right) = \alpha\|v\|_1^2$$

• L is bounded:

Trace ineq.

$$|L(v)| \leq \|f\| \|v\| + \|g\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \overset{\Downarrow}{\leq} \|f\| \|v\|_1 + \|g\|_{L^2(\Gamma)} \|v\|_1$$

$$= \left(\|f\|_{\infty} + \|g\|_{L^2(\Gamma)}\right) \|v\|_1$$

Thus, by Riesz: $\exists!\, u \in H^1(\Omega)$:

$$a(u,v) = L(v)$$.

Problem 3.7 (From the Book).

For
$$\mathcal{A}u = f \quad \text{in } \Omega \qquad f \in L^2(\Omega)$$
$$a\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma \qquad g \in L^2(\Gamma)$$

we saw that the variational formulation is:

Find $u \in H^1$ s.t.

$$a(u,v) = L(v) \qquad \forall v \in H^1.$$

where
$$a(u,v) = \int_\Omega a \nabla u \cdot \nabla v + c u v \, dx \qquad \begin{matrix} a \geq a_0 > 0 \\ c \geq c_0 > 0 \\ \text{smooth.} \end{matrix}$$
$$L(v) = \int_\Omega f v \, dx + \int_\Gamma g v \, ds.$$

we saw that this has a unique solution and proved the estimate

$$|L(v)| \leq C(\|f\| + \|g\|_{L^2(\Gamma)}) \|v\|_1 . \qquad (1)$$

Now the general energy estimate states:

$$\|u\|_V \leq C \|L\|_{V^*} \qquad (2)$$

Here $V = H^1$. From (1):

$$\frac{|L(v)|}{\|v\|_{H^1}} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)})$$

$$\Rightarrow \|L\|_{V^*} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)})$$
$$\overset{(2)}{\Rightarrow} \|u\|_{H^1} \leq C\|L\|_{V^*} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}).$$

Problem 3.8.

$$(P1)\begin{cases} -\nabla\cdot(a\nabla u)+cu=f & \text{in } \Omega \\ a\frac{\partial u}{\partial n}+h(u-g)=k & \text{on } \Gamma \end{cases}$$

$f\in C^2(\Omega)$, $g, k\in C^2(\Gamma)$, a, c, h smooth such that

$a(x)\geq a_0>0$, $c(x)\geq 0$ $x\in\Omega$, $h(x)\geq h_0>0$ $x\in\Gamma$.

Sol. Take $v\in C^1(\Omega)$, and suppose that $u\in C^2(\bar\Omega)$ is a classical solution of (P1). Then

$$\int_\Omega fv\,dx=\int_\Omega -\nabla\cdot(a\nabla u)v\,dx+\int_\Omega cuv\,dx$$

$$=-\int_\Gamma \underbrace{a\frac{\partial u}{\partial n}}_{=k-h(u-g)}v\,ds+\int_\Omega a\nabla u\cdot\nabla v\,dx+\int_\Omega cuv\,dx$$

$$=-\int(k-h(u-g))v\,ds+\int a\nabla u\cdot\nabla v\,dx+\int_\Omega cuv\,dx=-\int_\Gamma kv\,ds+\int_\Gamma huv\,ds$$

$$-\int_\Gamma hgv\,ds.$$

Weak formulation: Find $u\in H^1(\Omega)$ such that

$$\int_\Omega a\nabla u\cdot\nabla v\,dx+\int_\Omega cuv\,dx+\int_\Gamma huv\,ds=\int_\Omega fv\,dx+\int_\Gamma kv\,ds+\int_\Gamma hgv\,ds\quad\left(=\int_\Omega fv\,dx+\int_\Gamma(k+hg)v\,ds\right)$$

for all $v\in H^1(\Omega)$

Let $V=H^1(\Omega)$ and define

$$a(u,v)=\int_\Omega a\nabla u\nabla v\,dx+\int_\Gamma huv\,ds+\int_\Omega cuv\,dx$$

$$L(v)=\int_\Omega fv\,dx+\int_\Gamma(k+hg)v\,ds$$

(3)

To show:

(1) $a : V\times V \to \mathbb{R}$ is symmetric, bilinear, bounded, coercive.

(2) $l : V \to \mathbb{R}$ is linear bounded.

(1) (a) a is clearly symmetric, bilinear.

(b) bounded:

$$|a(u,v)| \leq |(a\nabla u, \nabla v)| + |(hu,v)_{L^2(\Gamma)}| + |(cu,v)|$$

Cauchy-Schwarz

$$\leq \|a\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|h\|_{L^\infty(\Gamma)} \|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} + \|c\|_{L^\infty} \|u\| \|v\|$$

Trace inequality for the second term

$$\leq \|a\|_{L^\infty} \|u\|_{H^1} \|v\|_{H^1} + C_T^2 \|h\|_{L^\infty(\Gamma)} \|u\|_{H^1} \cdot \|v\|_{H^1} + \|c\|_{L^\infty} \|u\| \|v\|$$

$$= \tilde{C} \|u\|_{H^1} \cdot \|v\|_{H^1} \qquad \tilde{C} = \left(\|a\|_{L^\infty(\Omega)} + C_T^2 \|h\|_{L^\infty(\Gamma)} + \|c\|_{L^\infty}\right)$$

(c) Coercive:

$$a(v,v) = \int_\Omega a|\nabla v|^2 dx + \int_\Gamma h|v|^2 ds + \int c\, v v\, dx$$

$$\geq a_0 \int_\Omega |\nabla v|^2 dx + h_0 \int_\Gamma |v|^2 ds$$

$$\geq \frac{1}{2} a_0 \int_\Omega |\nabla v|^2 dx + \frac{1}{2} a_0 \int_\Omega |\nabla v|^2 dx + h_0 \int_\Gamma v^2 ds$$

$$\geq \frac{1}{2} a_0 \int_\Omega |\nabla v|^2 dx + \min\left(\tfrac{1}{2} a_0, h_0\right) \left(\int_\Omega |\nabla v|^2 dx + \int_\Gamma v^2 ds\right)$$

Friedrich's inequality (Problem 3.4)

$$\geq \frac{1}{2} a_0 \int_\Omega |\nabla v|^2 dx + C \int_\Omega v^2 dx \geq K \|v\|_{H^1}^2, \quad K = \min\left(\tfrac{1}{2} a_0, C\right).$$

(2) (4)

Cauchy-Schwarz

$$|L(v)| \le |(f,v)| + |(k+h\cdot g, v)_{L^2(\Omega)}| \le \|f\|\|v\| + \|k+hg\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$

trace inequality for the second term

$$\le \|f\|\cdot\|v\|_{H^1} + C\|k+hg\|_{L^2(\Omega)}\|v\|_{H^1}$$

$$\le \|f\|\|v\|_{H^1} + C(\|k\|_{L^2(\Omega)} + \|h\|_{L^\infty}\|g\|_{L^2(\Omega)})\|v\|_{H^1}$$

$$= C_0\|v\|_{H^1}, \quad \text{where } C_0 = \|f\| + C(\|k\|_{L^2(\Omega)} + \|h\|_{L^\infty}\|g\|_{L^2(\Omega)})$$

Hence L is bounded.

This also implies that

$$\frac{|L(v)|}{\|v\|_{H^1}} \le \left(\|f\| + C\|k\|_{L^2(\Omega)} + \|h\|_{L^\infty}\|g\|_{L^2(\Omega)}\right)$$

$$\Rightarrow \|L\|_{V^*} \le C_1\left(\|f\| + \|k\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\right)$$

$\Rightarrow$ By Riesz, there is a unique $u \in H^1$:

$$a(u,v) = L(v) \quad \forall v \in H^1.$$

Energy estimate:

$$\|u\|_{H^1} = \|u\|_V \le c\|L\|_{V^*} \le C\left(\|f\|_{L^2(\Omega)} + \|k\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\right)$$

Weak formulation: Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in V.$$

(1) Let $a(u,v) = \int_\Omega \nabla u \nabla v \, dx$.

$L(v) = \int_\Omega f v \, dx$.

then

(1) a is symmetric, bilinear ✓

(2) a is bounded:

$$|a(u,v)| \le \|\nabla u\| \|\nabla v\| \le \|u\|_{H^1} \|v\|_{H^1}.$$

(3) a is coercive:

$$a(v,v) = \int_\Omega |\nabla v|^2 dx = \frac{1}{2}\|\nabla v\|^2 + \frac{1}{2}\|\nabla v\|^2 + \frac{1}{2}\Big(\underbrace{\int_\Omega v \, dx}_{=0}\Big)^2$$

Problem 3.5

$$\ge \frac{1}{2}\|\nabla v\|^2 + C\|v\|^2 = C_0 \|v\|^2_{H^1}, \quad C_0 = \min(\tfrac{1}{2}, C).$$

(4) L is bounded:

$$|L(v)| \le \|f\| \cdot \|v\| \le \|f\| \cdot \|v\|_{H^1}.$$

By Riesz: $\exists!\, u \in V$: $a(u,v) = L(v)$.

Problem 3.9. ⑥

$$-\Delta u = f \quad \text{in } \Omega \qquad f \in L^2(\Omega).$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma.$$

(a) Suppose that $u \in C^2(\bar\Omega)$ is a solution.
Take $v \in C^1(\bar\Omega)$. Then

$$\int_\Omega f v \, dx = \int_\Omega -\Delta u \, v \, dx = -\int_\Gamma \frac{\partial u}{\partial n} v \, ds + \int_\Omega \nabla u \cdot \nabla v \, dx$$
$$= \int \nabla u \nabla v \, dx.$$

Take $v \equiv 1$. Then

$$\int_\Omega f v \, dx = \int_\Omega f \, dx = \int_\Omega \nabla u \nabla v \, dx = 0$$

Hence $\int_\Omega f \, dx = 0$ is a necessary condition for existence.

(b) Let $V = \{ v \in H^1(\Omega) : \int_\Omega v \, dx = 0 \}$.
Note: V is a closed subspace of H^1:
$(v_n) \in V$ $v_n \to v$ in H^1, then as H^1 is complete
$v \in H^1$. Furthermore $v_n \to v$ in L^2 and hence
$$0 = \lim_{n\to\infty} \int v_n \, dx = \lim_{n\to\infty} (v_n, 1) = (\lim v_n, 1) = (v, 1)$$
$\Rightarrow v \in V$.
Therefore V itself is a Hilbert space with inner product and norm inherited from $H^1(\Omega)$.

(c) Suppose that u is a weak solution and that $u \in H^2$. Then, for all $v \in C^1(\bar\Omega)$, with $\int_\Omega v\,dx = 0$:

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx = -\int_\Omega \Delta u\, v\,dx + \int_\Gamma \frac{\partial u}{\partial n} v\,ds = \int_\Omega f v\,dx$$

$$\Rightarrow (*) \int_\Omega (-\Delta u - f) v\,dx + \int_\Gamma \frac{\partial u}{\partial n} v\,ds = 0 \quad \forall v \in C^1(\bar\Omega), \int_\Omega v\,dx = 0$$

$$\Rightarrow \int_\Omega (-\Delta u - f) v\,dx = 0 \qquad \forall v \in C_0^1(\Omega), \int_\Omega v\,dx = 0.$$

The problem is that $D = \{v \in C_0^1(\Omega), \int v\,dx = 0\}$ is not dense in L^2! It is only dense in $\{v \in L^2 : \int v\,dx = 0\}$.

Notice: $\int_\Omega \Big(\underbrace{\int_\Omega \Delta u\,dx}_{\frac{1}{m(\Omega)}\text{ constant}}\Big) \cdot v\,dx = 0$ and $\int_\Omega \Big(\underbrace{\int_\Omega f\,dx}_{\frac{1}{m(\Omega)}\text{ constant}}\Big) v\,dx = 0$

$$\Rightarrow \int_\Omega \Big[\Big(-\Delta u + \frac{1}{m(\Omega)}\int_\Omega \Delta u\,dx\Big) - \Big(f - \frac{1}{m(\Omega)}\int_\Omega f\,dx\Big)\Big] v\,dx = 0 \qquad \forall v \in D$$

(where $m(\Omega)$ is the Lebesgue measure of Ω)

Now $-\Delta u + \frac{1}{m(\Omega)}\int_\Omega \Delta u\,dx$ and $f - \frac{1}{m(\Omega)}\int_\Omega f\,dx$ are in $\{v \in L^2 : \int v\,dx = 0\}$

$$\Rightarrow -\Delta u + \frac{1}{m(\Omega)}\int_\Omega \Delta u\,dx - f - \frac{1}{m(\Omega)}\int_\Omega f\,dx = 0. \qquad (**)$$

Also notice, by integrating this against $v \in C^1(\bar\Omega)$, $\boxed{\int v\,dx = 0}$

$$\Rightarrow \int (-\Delta u - f) v\,dx = 0$$

$$\overset{(*)}{\Rightarrow} \int_\Gamma \frac{\partial u}{\partial n} v\,ds = 0 \qquad \forall v \in C^1(\bar\Omega), \int v\,dx = 0$$

But the restriction of such v to Γ is dense in $L^2(\Gamma)$ $\Rightarrow \frac{\partial u}{\partial n} = 0$ on Γ.

Finally:

$$\int_\Omega -\Delta u \cdot 1 \, dx = \int_\Gamma \frac{\partial u}{\partial n} \, ds = 0.$$

$$\Rightarrow \boxed{-\Delta u = f - \frac{1}{m(\Omega)} \int_\Omega f \, dx} \quad \text{by } (**)$$