(4.1) We will prove a more general statement and assume that $v : \mathbb{R} \to \mathbb{R}$ is continuous and bounded by a polynomial of degree N:

$$|v(x)| \le C(1+|x|^N) \qquad x \in \mathbb{R}$$

for some $N \geq 0$.

(a) We write

$$F(x,t) = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} v(y)\,dy = e^{-\frac{x^2}{4t}} \int_{-\infty}^{\infty} e^{\frac{xy}{2t}} e^{-\frac{y^2}{4t}} v(y)\,dy$$

$$:= e^{-\frac{x^2}{4t}} G(x,t).$$

For fixed x, t we have

$$\left| e^{\frac{xy}{4t}} e^{-\frac{y^2}{4t}} v(y) \right| \le e^{\frac{|x||y|}{4t}} \cdot |v(y)| \cdot e^{-\frac{y^2}{8t}} \cdot e^{-\frac{y^2}{8t}}$$
$$\le e^{\frac{|x||y|}{4t}} (1+|y|^N) e^{-\frac{y^2}{8t}} e^{-\frac{y^2}{8t}}$$
$$\le C_{x,t}\, e^{-y^2/8t}$$

Therefore the integral that defines G converges.

~~[illegible]~~ Let $h_n \to 0$, $|h_n| \le 1$.

Then, for fixed x and t,

$$\frac{G(x+h_n,t) - G(x,t)}{h_n} = \int_{-\infty}^{\infty} \underbrace{\frac{e^{(x+h_n)y/2t} - e^{xy/2t}}{h_n} e^{-y^2/4t} v(y)}_{f_n(y)}\,dy$$

$$|f_n(y)| = e^{\frac{xy}{2t}} \left| \frac{e^{h_n y/2t} - 1}{h_n} \right| e^{-y^2/4t} |v(y)|$$

$$= e^{\frac{xy}{2t}} e^{-y^2/4t} |v(y)| \left| \int_0^{h_n/2t} y e^{sy}\,ds \right| \cdot \frac{1}{|h_n|}$$

$$\le e^{\frac{|x||y|}{2t}} e^{-y^2/4t} C(1+|y|^N) \cdot \frac{|h_n|}{2t} |y| e^{\frac{|h_n||y|}{2t}} \cdot \frac{1}{|h_n|}$$

$|h_n|\le 1$

$$\le \frac{1}{2t} e^{\frac{|x||y|}{4t}} C(1+|y|^N) e^{\frac{1}{2t}|y|} |y| \cdot e^{-\frac{y^2}{8t}} \cdot e^{-\frac{y^2}{8t}}$$

$$\le C_{t,x}\, e^{-y^2/8t} \in L^1_n(\mathbb{R})$$

Furthermore $f_n(y) \to \frac{+y}{2t} e^{+xy/2t} \cdot e^{-y^2/2t} \cdot v(y) \ \forall y$.

By Lebesgue's dominated convergence theorem:

$$\lim_n \int f_n\, dy = \int \lim f_n\, dy = \int_{-\infty}^{\infty} \frac{+y}{2t} e^{+xy/2t} e^{-y^2/2t} v(y)\, dy$$

In particular $\frac{\partial G(x,t)}{\partial x}$ exists and

$$\frac{\partial G(x,t)}{\partial x} = \lim_{n\to\infty} \frac{G(x+h_n,t) - G(x,t)}{h_n} = \int_{-\infty}^{\infty} \frac{y}{2t} e^{\frac{xy}{2t}} e^{-\frac{y^2}{2t}} v(y)\, dy$$

Therefore $\frac{\partial F}{\partial x}$ exists and, by the product rule,

$$\frac{\partial F}{\partial x} = e^{-\frac{x^2}{4t}} \cdot \left(-\frac{x}{2t}\right) \int_{-\infty}^{\infty} e^{\frac{xy}{2t}} e^{-\frac{y^2}{2t}} v(y)\, dy$$

$$+ e^{-\frac{x^2}{2t}} \int_{-\infty}^{\infty} \frac{y}{2t} e^{\frac{xy}{2t}} e^{-\frac{y^2}{2t}} v(y)\, dy$$

$$= -\int_{-\infty}^{\infty} \frac{(x-y)}{2t} e^{-\frac{(x-y)^2}{4t}} v(y)\, dy.$$

(b) Note that

$$\frac{\partial F}{\partial x} = -\frac{x}{2t} F(x,t) + \frac{1}{2t}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y\, v(y)\, dy.$$

If $|v(y)| \le C(1+|y|^N)$, then $|y\,v(y)| \le C(1+|y|^{N+1})$. Hence by the previous part and the product rule $\frac{\partial F}{\partial x}$ is differentiable and

$$\frac{\partial^2 F}{\partial x^2} = +\frac{x}{2t}\int_{-\infty}^{\infty} \frac{x-y}{2t} e^{-\frac{(x-y)^2}{4t}} v(y)\,dy$$

$$-\frac{1}{2t}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} v(y)\,dy$$

$$-\frac{1}{2t}\int \frac{(x-y)}{2t} e^{-\frac{(x-y)^2}{4t}} \cdot y\,v(y)\,dy$$

$$= \frac{1}{2t}\int_{-\infty}^{\infty} \frac{x(x-y)-2t-y(x-y)}{2t} e^{-\frac{(x-y)^2}{4t}} v(y)\,dy$$

$$= \frac{1}{2t}\int_{-\infty}^{\infty} \frac{(x-y)^2-2t}{2t} e^{-\frac{(x-y)^2}{4t}} v(y)\,dy.$$

4.2 Let $G(x,r) = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4}r} v(y)\,dy$ and fix $r>0$ and $x \in \mathbb{R}$.

Let $h_n \to 0$, $|h_n| \le r/2$. Then

$$\frac{G(x,r+h_n)-G(x,r)}{h_n} = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4}\cdot r} \frac{e^{-\frac{(x-y)^2}{4}\cdot h_n}-1}{h_n} v(y)\,dy$$

Let $f_n(y) = e^{-\frac{(x-y)^2}{4}r} \frac{e^{-\frac{(x-y)^2}{4}h_n}-1}{h_n} v(y)$. Then

$$|f_n(y)| \le e^{-\frac{(x-y)^2 r}{4}} |v(y)| \left|\frac{1}{h_n}\int_0^{h_n} -\frac{(x-y)^2}{4} e^{-\frac{(x-y)^2}{4}s} ds\right|$$

$$\le e^{-\frac{(x-y)^2 r}{4}} C(1+|y|^N) \cdot \frac{1}{|h_n|} \cdot |h_n| \cdot \frac{(x-y)^2}{4} \cdot 1\, e^{\frac{(x-y)^2}{4}\underbrace{|h_n|}_{\le r/2}}$$

$$\le e^{-\frac{(x-y)^2 r}{16}} C(1+|y|^N)\,\frac{(x+y)^2}{4}\, e^{-\frac{(x-y)^2 r}{16}} \le C_{r,x}\, e^{-\frac{(x-y)^2 r}{16}} \in L^1(\mathbb{R})\ (\text{in } y)$$

$$(=: C_{r,x} f_{x,r}(y))$$

(Remember x, r fixed!)

Also

$$f_n(y) \to e^{-\frac{(x-y)^2 r}{4}}\, \frac{-(x-y)^2}{4}\, v(y). \quad \forall y.$$

as $n \to \infty$.

By Lebesgue's dominated convergence theorem:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n\, dy = \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n\, dy = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2 r}{4}}\, \frac{-(x-y)^2}{4}\, v(y)\, dy$$

Thus G is differentiable in r and

$$\frac{\partial G}{\partial r} = \lim_{n\to\infty} \frac{G(x, r+h_n) - G(x,r)}{h_n} = -\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2 r}{4}}\, \frac{(x-y)^2}{4}\, v(y)\, dy.$$

As $F(x,t) = G(x, \frac{1}{t})$, F is differentiable in t by the ~~chain~~ composition theorem and:

$$\frac{\partial F}{\partial t} = \frac{\partial G}{\partial r}(x, \tfrac{1}{t}) \cdot -\frac{1}{t^2} = \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\, \frac{(x-y)^2}{4t^2}\, v(y)\, dy.$$

(4.3) Just combine (4.1) and (4.2) and use the product rule.

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(4.4.) Using (4.1) and (4.2) (with the general v that is polynomially bounded) one easily sees that

$$\frac{\partial^3 u}{\partial x \partial t^2} = \frac{\partial^3 u}{\partial t^2 \partial x} \text{ exists.}$$

and we may differentiate under the integral sign. Then

$$\frac{\partial^3 F}{\partial x \partial t^2} = \int_{-\infty}^{\infty} v(y) \frac{\partial^3 u}{\partial x \partial t^2}\left(e^{-\frac{(x-y)^2}{4t}}\right) dy$$

We have:

$$\frac{\partial e^{-\frac{(x-y)^2}{4t}}}{\partial x} = -\frac{(x-y)}{4t} e^{-\frac{(x-y)^2}{4t}}$$

$$\frac{\partial}{\partial t}(\quad) = \frac{(x-y)}{4t^2} e^{-\frac{(x-y)^2}{4t}} - \frac{(x-y)^3}{16t^3} e^{-\frac{(x-y)^2}{4t}}$$

$$\frac{\partial^2}{\partial t}(\quad) = \frac{-(x-y)}{2t^3} e^{-\frac{(x-y)^2}{4t}} + \frac{(x-y)^3}{16t^4} e^{-(x-y)^2/4t} + \frac{3}{16}\frac{(x-y)^3}{t^4} e^{-\frac{(x-y)^2}{4t}}$$

$$- \frac{(x-y)^5}{64t^5} e^{-\frac{(x-y)^2}{4t}}$$

Let $\tilde{u} = (4\pi t^{-1/2}) F$

$$\frac{\partial \tilde{u}}{\partial x} = (4\pi t)^{-1/2} \frac{\partial F}{\partial x}$$

$$\frac{\partial^2 \tilde{u}}{\partial t \partial x} = -\frac{(2\pi)^{1/2}}{2} t^{-3/2} \frac{\partial F}{\partial x} + (4\pi t)^{-1/2} \frac{\partial^2 F}{\partial t \partial x}$$

$$\frac{\partial^3 \tilde{u}}{\partial t^2 \partial x} = C_1 t^{-5/2} \frac{\partial F}{\partial x} + C_2 t^{-3/2} \frac{\partial^2 F}{\partial t \partial x} + C_3 t^{-3/2} \frac{\partial^2 F}{\partial t \partial x} + C_4 t^{-1/2} \frac{\partial^3 F}{\partial t^2 \partial x}$$

Note: If P is a polynomial, then

$$\left| P\left(\left(\frac{x-y}{\sqrt{t}}\right)^2\right) e^{-\left(\frac{x-y}{\sqrt{t}}\right)^2 \frac{1}{8}} \right| \le M, \text{ where } M$$

only depends on the coefficients of the polynomial P. Therefore,

$$\left|\frac{\partial^3 \tilde{u}}{\partial t^2 \partial x}(x,t)\right| \le C_1 t^{-3} e^{-\frac{(x-y)^2}{8t}} + C_2 t^{-3} e^{-\frac{(x-y)^2}{8t}} + C_3 t^{-3} e^{-\frac{(x-y)^2}{8t}}$$

$$= \tilde{C}\, t^{-3} e^{-\frac{(x-y)^2}{8t}}$$

$$\Rightarrow \left|\frac{\partial u(x,t)}{\partial t^2 \partial x}\right| \le \tilde{C} \int_{-\infty}^{\infty} t^{-3} |v(y)|\, e^{-\frac{(x-y)^2}{8t}}\, dy$$

$$\le \tilde{C}\, t^{-3} \sup_{y\in\mathbb{R}} |v(y)| \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{8t}}\, dy = \left| \begin{array}{l} \frac{-x+y}{\sqrt{t}} = s \\ dy = \sqrt{t}\, ds \end{array} \right|$$

$$= \tilde{C}\, t^{-3}\, \|v\|_C \int_{-\infty}^{\infty} e^{-\frac{s^2}{8}} \sqrt{t}\, ds$$

$$= \tilde{\tilde{C}}\, t^{-5/2}\, \|v\|_C$$

$$\Rightarrow$$

$$\|D^\alpha u(\cdot,t)\|_{C(\mathbb{R})} \le \tilde{\tilde{C}}\, t^{-5/2}\, \|v\|_{C(\mathbb{R})}.$$

(4.5) $-\Delta u = f$ in Ω $\Omega = \{|x| < R\}$ (7)

$u = g$ on Γ

$f(x) = h(|x|)$ $g = g(\varphi)$.

(1) Solve $-\Delta u_1 = 0$ in Ω

$u_1 = g$ on Γ

Poisson's formula:

$$u_1(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} P_R(r, \varphi - \psi)\, g(\psi)\, d\psi \quad , P_R(r,\varphi) = \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos\varphi}$$

(2) Solve $-\Delta u_2 = f$ in Ω

$u_2 = 0$ on Γ

$x = (r\cos\varphi, r\sin\varphi)$

$$u_2(x) = \int_\Omega G(x,y) f(y)\, dy = \int_0^{2\pi}\int_0^R G((r\cos\varphi, r\sin\varphi), (r'\cos\varphi', r'\sin\varphi'))\, h(r')\, r'\, dr'\, d\varphi'$$

$$G(x,y) = U(x-y) - v_y(x) = -\frac{1}{2\pi}\log|x-y| - v_y(x)$$

$v_y(x)$: $-\Delta_x v_y(x) = 0$ $x \in \Omega$

$v_y(x) = U(x-y)$ $x \in \Gamma$

$$v_{(r'\cos\varphi', r'\sin\varphi')}(r\cos\varphi, r\sin\varphi) = -\frac{1}{4\pi}\int_0^{2\pi} P_R(r, \varphi - \psi) \log\left[(R\cos\psi - r'\cos\varphi')^2 + (R\sin\psi - r'\sin\varphi')^2\right]^{1/2} d\psi$$

$\Rightarrow u_2(r\cos\varphi, r\sin\varphi) =$

$$\int_0^{2\pi}\int_0^R \Big\{ -\frac{1}{2\pi}\log\left[(r\cos\varphi - r'\cos\varphi')^2 + (r\sin\varphi - r'\sin\varphi')^2\right]^{1/2} + \frac{1}{4\pi}\int_0^{2\pi} P_R(r,\varphi-\psi)\log\left[(R\cos\psi - r'\cos\varphi')^2 + (R\sin\psi - r'\sin\varphi')^2\right]^{1/2} d\psi \Big\} \cdot h(r')\, r'\, dr'\, d\varphi'$$

Finally $u = u_1 + u_2$.