(5.1)

$$-\Delta u + u^3 = 0 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \Gamma.$$

Then

$$\int_\Omega -\Delta u \, u \, dx + \int_\Omega u^3 u \, dx = 0$$

$$-\underbrace{\int \frac{\partial u}{\partial n} u \, ds}_{0} + \int_\Omega |\nabla u|^2 dx + \int u^4 dx = 0$$

$$\int |\nabla u|^2 dx + \int u^4 dx = 0$$

$$\Rightarrow \quad u = 0.$$

(5.2) If $u \le 0$ on $\Omega \times (0,T)$, then there is nothing to prove. Suppose that there is $(x_0, t_0) \in \Omega \times (0,T)$ such that $u(x_0, t_0) > 0$.

Suppose, by contradiction, that u attains its maximum at $(\tilde{x}, \tilde{t}) \in \Omega \times (0,T]$. Then

$$u(\tilde{x}, \tilde{t}) > 0.$$

By assumption

$$\partial_t u(\tilde{x},\tilde{t}) - \Delta u(\tilde{x},\tilde{t}) + u(\tilde{x},\tilde{t}) \le 0 \qquad (*)$$

(by continuity this holds on $\Omega \times (0,T]$ not just on $\Omega \times (0,T)$).

2

On the other hand

$$-\Delta u(\tilde{x},\tilde{t}) \geq 0 \qquad \text{as } \tilde{x} \text{ is an}$$

interior point of Ω and u has it's maximum there ("concave down")

Furthermore, $u_t(\tilde{x},\tilde{t}) = 0$ if $\tilde{t} < T$

($\tilde{t}$ is an interior point of $(0,T)$ and hence the gradient is 0 there)

Or, $u_t(\tilde{x},\tilde{t}) \geq 0$ if $\tilde{t} = T$

(the maximum is at the right end point of $(0,T]$).

Since $u(\tilde{x},\tilde{t}) > 0$, we have:

$$u_t(\tilde{x},\tilde{t}) - \Delta u(\tilde{x},\tilde{t}) + c(\tilde{x})u(\tilde{x},\tilde{t}) > 0.$$

This contradicts (*).

5.3

$$\partial_t u - \partial_x^2 u = (x^2-1)u \qquad \text{in } (0,1)\times(0,\infty)$$

$$u(0,t) = u(1,t) = 0 \qquad \text{for } t > 0.$$

$$u(x,0) = \sqrt{x(1-x)} \qquad x \in [0,1].$$

(a) Multiply the equation by u and integrate

$$\int_0^1 u\,\partial_t u\,dx - \underbrace{\int_0^1 u\,\partial_x^2 u\,dx}_{\underbrace{[u\partial_x u]_0^1}_{0} + \int_0^1 (\partial_x u)^2 dx} = \int_0^1 u^2 \underbrace{(x^2-1)}_{\leq 0}\,dx \leq 0$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\partial_x u\|^2 \leq 0$$

$$\Rightarrow \quad \frac{1}{2}\|u(t)\|^2 - \frac{1}{2}\|u(0)\|^2 + \int_0^t \|\partial_x u(s)\|^2\,ds \leq 0$$

$$\Rightarrow \frac{1}{2}\|u(t)\|^2 + \int_0^t |u(s)|_1^2\,ds \leq \frac{1}{2}\|u(0)\|^2$$

$$\Rightarrow \cancel{\tfrac{1}{2}}\|u(t)\|^2 \leq \cancel{\tfrac{1}{2}}\|u(0)\|^2 = \int_0^1 x(1-x)\,dx = \frac{1}{6} \quad . \quad (t \geq 0).$$

(4) By problem 5.3:

$$u(x,t) \leq \max(0, \max_{\Gamma_p} u) \leq \max\Big(0, \max_{x\in[0,1]} \sqrt{x(1-x)}\Big)$$

$$\leq \frac{1}{2}$$

5.4 $\quad e^x \partial_x u + \partial_y u = 0 \qquad u(0,y) = y^4 \qquad \Gamma_I$

characteristics:

$$\dot{x}(s) = e^x \qquad \dot{y}(s) = 1$$

$$\Rightarrow e^{-x}\dot{x} = 1 \qquad y(s) = s + K$$

$$-\frac{d}{ds}e^{-x} = 1$$

$$-e^{-x} = s + c = y + D$$

Characteristics: $(t ; D - e^{-t})$

characteristics through $(\bar{x}, \bar{y})$:

$$t = \bar{x}$$

$$-D - e^{-t} = \bar{y} \Rightarrow -D e^{-\bar{x}} - \bar{y}$$

$$-D = \bar{y} + e^{-\bar{x}}$$

$$\Rightarrow \quad \zeta(t) = \left(t, \bar{y} + e^{-\bar{x}} - e^{-t} \right)$$

at $t = 0$ $\zeta(0) \in \Gamma_r$.

$$\Rightarrow \quad u(\bar{x}, \bar{t}) = v(\zeta(0)) = \left(\bar{y} + e^{-\bar{x}} - 1 \right)^4.$$

5.5 Consider $\bar{u}(x, y, t) = u(x, t)$. If

$$\partial_t^2 u - \Delta u = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}_+$$

$$u(x, 0) = v(x)$$

$$\partial_t u(x, 0) = w(x).$$

then

$$\partial_t^2 \bar{u} - \Delta \bar{u} = 0 \qquad \text{in } \mathbb{R}^2 \times \mathbb{R}_+$$

$$\bar{u}(x, y, 0) = \bar{v}(x, y) := v(x) \qquad (x, y) \in \mathbb{R}^2$$

$$\partial_t \bar{u}(x, y, 0) = \bar{w}(x, y) := w(x) \qquad (x, y) \in \mathbb{R}^2$$

Let $\bar{x} = (x, 0)$. Then

$$u(x, t) = \bar{u}(\bar{x}, t) = \frac{\partial}{\partial t} \frac{1}{2\pi} \int_{D_t(\bar{x})} \frac{\bar{v}(y)}{\sqrt{t^2 - |\bar{x} - y|^2}} \, dy + \frac{1}{2\pi} \int_{D_t(\bar{x})} \frac{w(y)}{\sqrt{t^2 - |\bar{x} - y|^2}} \, dy$$

Now

$$\int_{D_t(\bar{x})} \frac{v(y)}{\sqrt{t^2-|\bar{x}-y|^2}}\,dy = \int_{x-t}^{x+t} \int_{-\sqrt{t^2-(y_1-x)^2}}^{\sqrt{t^2-(y_1-x)^2}} \frac{v(y_1)}{\sqrt{t^2-(x-y_1)^2-y_2^2}}\,dy_2\,dy_1$$

$$= \int_{x-t}^{x+t} v(y_1) \int_{-\sqrt{t^2-(y_1-x)^2}}^{\sqrt{t^2-(y_1-x)^2}} \frac{1}{\sqrt{t^2-(x-y_1)^2-y_2^2}}\,dy_2\,dy_1$$

we have, with $a := t^2-(x-y_1)^2$

$$\int_{-\sqrt{a}}^{\sqrt{a}} \frac{1}{\sqrt{a}} \frac{1}{\sqrt{1-\left(\frac{y_2}{\sqrt{a}}\right)^2}}\,dy_2 = \left| \begin{array}{l} s = \frac{y_2}{\sqrt{a}} \\ ds = \frac{1}{\sqrt{a}}\,dy_2 \end{array} \right|$$

$$= \int_{-1}^{1} \frac{1}{\sqrt{1-s^2}}\,ds = \sin^{-1}(1) - \sin^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi .$$

Thus

$$u(x,t) = \frac{\partial}{\partial t} \frac{1}{2} \int_{x-t}^{x+t} v(y_1)\,dy_1 + \frac{1}{2} \int_{x-t}^{x+t} w(y_2)\,dy_2$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2}\int_0^{x+t} v(y_1)\,dy_1 - \int_0^{x-t} v(y_1)\,dy_1\right) + \frac{1}{2}\int_{x-t}^{x+t} w(y_2)\,dy_2$$

Fundamental thm of calculus + chain rule

$$\overset{\downarrow}{=} \boxed{\frac{1}{2}\left(v(x+t) + v(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} w(y)\,dy}$$

5-6 Let $\phi \in C_0^\infty$. Then

6

$$(x^2+x)\delta'(\phi) = \delta'((x^2+x)\phi) = -\frac{d}{dx}\left((x^2+x)\phi(x)\right)\Big|_{x=0}$$

$$= -(2x+1)\phi(x)\Big|_{x=0} - (x^2+x)\phi'(x)\Big|_{x=0} = -\phi(0) = -\delta(\phi)$$

thus $a=-1$.