TMA690 Partial Differential Equations - Exam

Date: Thursday, 11 January
Time: 8.30-12.30

- Please write your answers to SECTION A on the actual exam paper and hand it in together with your solution to problems in SECTION B.
- Telephone contact during the exam: Anders Hildeman, ext. 5325.
- The final exam is worth 50 points. 20 points are required for a pass grade 3, 30 points for grade 4, and 40 points for grade 5. The bonus points gathered during the course are added to your exam total.
- Solutions will be announced after the end of the exam on the course homepage.
- You may view the exams on Friday, January 19, 10.30-11.30 in MV:L15.

SECTION A

Indicate whether the following statements are True or False by putting T or F in the box. You will get 1 point for each right answer.

1. The linear space V of continuous functions $f : [0,1] \to \mathbb{R}$ equipped with the norm $\|f\| = \left(\int_0^1 |f(s)|^2 \, \mathrm{d}s\right)^{1/2}$ is a Banach space. $\boxed{F}$

2. The space $H_0^1(0,1)$ is a closed subspace of $H^1(0,1)$. $\boxed{T}$

3. The weak derivative $\frac{\partial v}{\partial x_2}$ of a function $v = v(x_1, x_2) \in L^2((0,1) \times (0,1))$ is a function that belongs to $L^2((0,1) \times (0,1))$. $\boxed{F}$

4. The bilinear form $a : H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ defined by
$$a(u,v) = \int_0^1 u'v'dx,$$
where u' and v' are the generalized derivatives of u and v is coercive. $\boxed{F}$

5. Functions that belong to $H^1((0,1) \times (0,1))$ are continuous. $\boxed{F}$

6. Let $\Omega = (0,1) \times (0,1)$ and let Γ denote its boundary. Let $a(x,y) = 1 + x^2 + y^2$. The equation
$$-\nabla \cdot (a \nabla u) = 0, \text{ in } \Omega,$$
$$u|_\Gamma = 1$$
has a unique weak solution. $\boxed{T}$

7. If u is twice continuously differentiable and $-u'' + u \leq 0$ on $(0,1)$, then
$$\max_{x \in [0,1]} u(x) \leq u(1).$$
$\boxed{F}$

Please turn over...

8. If u_1 and u_2 are classical solutions of the initial value problems

$$\frac{\partial u_i}{\partial t} - \Delta u_i = 0 \text{ in } \mathbb{R}^3 \times \mathbb{R}_+, \quad i = 1, 2,$$
$$u_i(x, y, z, 0) = ie^{-x^2-y^2-z^2} \quad i = 1, 2,$$

then $|u_1(x, y, z, t) - u_2(x, y, z, t)| \leq 1$ in $\mathbb{R}^3 \times \mathbb{R}_+$.

T

9. If u is a sufficiently smooth solution of

$$u_{tt} - u_{xx} = 0 \text{ in } (0, 1) \times (0, \infty)$$
$$u(0, t) = u(1, t) = 0, \quad t > 0,$$
$$u(x, 0) = x - x^2, \quad x \in (0, 1),$$
$$u_t(x, 0) = x, \quad x \in (0, 1),$$

then $\int_0^1 (u_t(x, t))^2 + (u_x(x, t))^2 \, \mathrm{d}x = \frac{2}{3}$ for all $t \geq 0$.

T

10. The only solution of the initial value problem

$$u_t - \Delta u = 0 \text{ in } \mathbb{R}^3 \times (0, \infty)$$
$$u(x, y, z, 0) = 0,$$

is the function $u(x, y, z, t) = 0$.

F

SECTION B

Solve the following problems and hand in your detailed answers.

11. (a) State the Riesz Representation Theorem. (2)
    (b) State the Lax-Milgram Lemma. (2)

12. (a) Let $\Omega \subset \mathbb{R}^d$ is a bounded domain with appropriately smooth boundary. Define the weak derivative $\frac{\partial v}{\partial x_i}$ $(i = 1, 2.., d)$ of a function $v \in L^2(\Omega)$ and explain when does $\frac{\partial v}{\partial x_i}$ belong to $L^2(\Omega)$. (3)
    (b) Define the Sobolev space $H^1(\Omega)$. (2)

13. State and prove Friedrich's inequality. (8)

14. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ. Give a weak (variational) formulation of the problem

$$-\Delta u = 1 \text{ in } \Omega$$

$$\frac{\partial u}{\partial n} + u = 0 \text{ on } \Gamma.$$

Prove existence and uniqueness of a weak solution. (8)

15. (a) Define the space of test functions on $\mathbb{R}$ and the convergence of test functions. (1)
    (b) Define distribution (generalized function) on $\mathbb{R}$. (1)
    (c) Define the derivative of a distribution on $\mathbb{R}$. (1)
    (d) Define the weak convergence of distributions on $\mathbb{R}$. (1)
    (e) Let $g(x) = \frac{3}{4}(1 - x^2)$ for $|x| \leq 1$ and $g(x) = 0$ for $|x| > 1$. Show that if $g_\varepsilon(x) = \frac{1}{\varepsilon}g(\frac{1}{\varepsilon}x)$ then the regular distribution u_{g_ε} associated with g_ε converges weakly to δ as $\varepsilon \to 0$, where δ denotes the Dirac delta distribution (concentrated at 0).
    (4)

16. Solve the partial differential equation

$$x\partial_x u + 3y\partial_y u = 0,$$

subject to the boundary condition $u(1, y) = y^2$. (7)

SECTION B

(11) (a) Let V be a Hilbert space with scalar product $(\cdot,\cdot)$. For each bounded linear functional L on V there is a unique $u \in V$ such that

$$L(v) = (v, u) \quad \forall v \in V.$$

Furthermore, $\|L\|_{V^*} = \|u\|_V$.

(b) Let V be a Hilbert space. If the bilinear form $a(\cdot,\cdot)$ is bounded and coercive on V and L is a bounded linear functional on V, then there is a unique vector $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V.$$

Furthermore, there is $C > 0$, such that

$$\|u\|_V \leq C \|L\|_{V^*}.$$

(12) (a) Let $v \in L^2$. The weak derivative of v $\left(\frac{\partial v}{\partial x_i}\right)$ is ~~the~~ linear functionale:

$$\frac{\partial v}{\partial x_i}(\varphi) = -\int_\Omega v \frac{\partial \varphi}{\partial x_i}\, dx \qquad \forall \varphi \in C_0^1(\Omega).$$

If there is a constant $C > 0$ such that

$$\left|\frac{\partial v}{\partial x_i}(\varphi)\right| \leq C \|\varphi\|_{L^2} \qquad \forall \varphi \in C_0^1(\Omega)$$

then, as C_0^1 is dense in $L^2(\Omega)$, the functional $\frac{\partial v}{\partial x_i}$ can be extended to a bounded linear functional defined on $L^2(\Omega)$, still denoted by $\frac{\partial v}{\partial x_i}$. ②

In this case, by the Riesz Representation Theorem, there is $w \in L^2(\Omega)$ such that

$$\frac{\partial v}{\partial x_i}(\varphi) = \int_\Omega w \varphi \, dx .$$

We then say that the weak derivative $\frac{\partial v}{\partial x_i}$ belongs to $L^2(\Omega)$ and $\frac{\partial v}{\partial x_i} = w$.

(b) The space $H^1(\Omega)$ consists on functions $v \in L^2(\Omega)$ such that $\frac{\partial v}{\partial x_i} \in L^2(\Omega)$ for all $i = 1, \dots, d$. It is a Hilbert space with inner product

$$(v,w)_{H^1} = (v,w)_{L^2} + (\nabla u, \nabla w)_{L^2}$$
$$= \int_\Omega u v \, dx + \sum_{i=1}^{d} \int \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_i} \, dx .$$

(13) Friedrich's inequality:

$$\|v\|_{L^2(\Omega)} \le C \left(\|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Gamma)} \right)^{1/2} \quad v \in C^1$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with appropriately smooth boundary.

<u>Proof</u>. Let $\phi = \frac{1}{2d} |x|^2$, then $\Delta \phi = 1$. Therefore,

$$\int_\Omega v^2 \, dx = \int v^2 \Delta \phi \, dx = \int_\Gamma v^2 \nabla \phi \cdot n \, ds - 2 \int v \nabla v \cdot \nabla \phi \, dx$$

$$\Rightarrow \|v\|^2 \le \int_\Gamma v^2 |\nabla \phi| ds + 2 \int_\Omega |v| |\nabla v| |\nabla \phi| \, dx$$

since Ω is bounded, there is $M > 0$ such that

$\max_{\bar{\Omega}} |D\phi| < M$. Thus

$$\|v\|^2 \le M \int_\Gamma v^2 ds + 2M \int |v||\nabla v| dx$$

$$\le M \|v\|^2_{L^2(\Gamma)} + \int_\Omega \frac{|v|^2}{2} + \frac{(2M|\nabla v|)^2}{2} dx$$

$$= M \|v\|^2_{L^2(\Gamma)} + \frac{1}{2}\|v\|^2 + 2M^2 |v|_1^2$$

$$\Rightarrow \frac{1}{2}\|v\|^2 \le M \|v\|^2_{L^2(\Gamma)} + 2M^2 |v|_1^2$$

$$\Rightarrow \|v\|^2_{L^2(\Omega)} \le C\left(\|v\|_{L^2(\Gamma)} + \|\nabla v\|^2_{L^2(\Omega)}\right) \quad \square.$$

14 $\quad -\Delta u = 1 \quad$ in Ω

$\frac{\partial u}{\partial n} + u = 0 \quad$ on Γ.

Let $v \in C^1(\Omega)$. then

$$\int_\Omega 1 \cdot v\, dx = \int_\Omega -\Delta u\, v\, dx = -\int_\Gamma \frac{\partial u}{\partial n} v\, ds + \int_\Omega \nabla u \nabla v\, dx$$

$$= \int_\Gamma u v\, ds + \int_\Omega \nabla u \cdot \nabla v\, dx$$

Weak formulation: Find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Gamma u v\, ds = \int_\Omega v\, dx \qquad \forall v \in H^1(\Omega).$$

Let $V = H^1(\Omega)$. We Define

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Gamma u v\, ds$$

$$L(v) = \int_\Omega v\, dx$$

(4)

- a is bilinear, symmetric ✓
- a is bounded:

$$|a(u,v)| \le \|\nabla u\| \, \|\nabla v\| + \|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)}$$

Trace inequality.

$$\le \|u\|_1 \cdot \|v\|_1 + C^2 \|u\|_1 \|v\|_1$$

$$= (1+C^2) \|u\|_1 \|v\|_1$$

- a is coercive:

$$a(v,v) = |v|_1^2 + \|v\|^2_{L^2(\Gamma)} \ge |v|_1^2 + \frac{1}{2}\|v\|^2_{L^2(\Gamma)} = \frac{1}{2}|v|_1^2 + \frac{1}{2}\left(|v|_1^2 + \|v\|^2_{L^2(\Gamma)}\right)$$

$$\ge \frac{1}{2}|v|_1^2 + C\|v\|^2_{L^2(\Omega)} \ge \min\left(\tfrac{1}{2}, C\right)\left(|v|_1^2 + \|v\|^2\right) = \alpha \|v\|_1^2 .$$

Friedrich's ineq.

- L is bounded:

$$|L(v)| = \left|\int_\Omega 1 \cdot v \, dx\right| \le \left(\int_\Omega 1^2 dx\right)^{1/2} \cdot \left(\int v^2 dx\right)^{1/2} = C \|v\|^2$$

$$\le C \|v\|_1^2$$

therefore, by Riesz (or Lax-Milgram), there is a unique $u \in H^1(\Omega)$ such that

$$a(u,v) = L(v).$$

(15) (a) The space of test functions C_0^∞ consists of functions $\phi: \mathbb{R} \to \mathbb{R}$ such that

(a) $\phi^{(m)}$ exists for all $m = 1, 2, \dots$

(b) $\phi = 0$ outside of a bounded and closed interval.

Convergence of test functions:

$\phi_j \to \phi$ if

(a) ϕ and ϕ_j vanish outside of a common bounded and closed interval

(b) if $\phi_j^{(m)} \to \phi^{(m)}$ uniformly for all $m = 0, 1, \ldots$

(b) A distribution u is a linear functional on C_0^∞ that is continuous with respect to the convergence of test functions:

If $\phi_j \to \phi$ then $u(\phi_j) \to u(\phi)$.

(c) Let u be a distribution. Then u' is a distribution defined by $u'(\phi) = -u(\phi') \quad \forall \phi \in C_0^\infty$.

(d) A sequence of distribution (u_j) converges to a distribution u weakly if

$$u_j(\phi) \to u(\phi) \qquad \forall \phi \in C_0^\infty.$$

(e) To show: $u_{g_\varepsilon}(\phi) \to \phi(0) \quad \forall \phi \in C_0^\infty$.

Note: $g_\varepsilon \geq 0$, $\int_{\mathbb{R}} g_\varepsilon \, dx = 1 = \int_{-1}^{1} g(x)\, dx$

$$|u_{g_\varepsilon}(\phi) - \phi(0)| = \left| \int_{\mathbb{R}} g_\varepsilon(x)\phi(x)\,dx - \int_{\mathbb{R}} g_\varepsilon(x)\phi(0)\,dx \right|$$

$$\leq \int_{\mathbb{R}} g_\varepsilon(x)|\phi(x) - \phi(0)|\,dx = \int_{-1}^{1} g(x)|\phi(\varepsilon x) - \phi(0)|\,dx$$

$$\leq \sup_{x \in [-1,1]} |\phi(\varepsilon x) - \phi(0)| = \sup_{z \in [-\varepsilon, \varepsilon]} |\phi(z) - \phi(0)| \to 0$$

as $\varepsilon \to 0$ because ϕ is continuous at 0.

(26) characteristic equations:

$$\dot{x} = x \qquad \dot{y} = 3y$$

$\Rightarrow x(t) = Ce^t \qquad y(t) = De^{3t}$

$\Rightarrow \xi(t) = (Ce^t, De^{3t})$ or with $s = Ce^t$,

$\xi(s) = (s, Ks^3)$ or with $s = r+1$

$\xi(r) = (r+1, K(r+1)^3)$. Then $\xi(0) = (1, K) \in \Gamma$.

~~Let~~ Need: $\left.\begin{array}{l} \cancel{r}+1 = \tilde{x} \\ K(r+1)^3 = \tilde{y} \end{array}\right\} \quad K = \dfrac{\tilde{y}}{\tilde{x}^3}$

$\Rightarrow \xi(r) = \left(r+1, \dfrac{\tilde{y}}{\tilde{x}^3}(r+1)\right)$

$u(\tilde{x}, \tilde{y}) = v(\xi(0)) = \left(\dfrac{\tilde{y}}{\tilde{x}^3}\right)^2$

The solution is $u(x,y) = y^2 \cdot x^{-6}$

check: $u(1,y) = y^2$

$\left.\begin{array}{l} x\partial_x u = -6y^2x^{-6} \\ 3y\partial_y u = 6y^2x^{-6} \end{array}\right\} \quad x\partial_x u + 3y\partial_y u = 0.$